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Propagation of Planetary-Scale Disturbances from the Lower into the Upper Atmosphere

J. G. CHARNEY AND P. G. DRAZIN[1]

Massachusetts Institute of Technology
Cambridge, Massachusetts

Abstract. The possibility that a significant part of the energy of the planetary-wave disturbances of the troposphere may propagate into the upper atmosphere is investigated. The propagation is analogous to the transmission of electromagnetic radiation in heterogeneous media. It is found that the effective index of refraction for the planetary waves depends primarily on the distribution of the mean zonal wind with height. Energy is trapped (reflected) in regions where the zonal winds are easterly or are large and westerly. As a consequence, the summer circumpolar anticyclone and the winter circumpolar cyclone in the upper stratosphere and mesosphere are little influenced by lower atmosphere motions. Energy may escape into the mesosphere near the equinoxes, when the upper-atmosphere zonal flow reverses. At these times tunneling of the energy through a reflecting barrier is also possible. Most of the time, however, there appears to be little mechanical coupling on a planetary scale between the upper and lower atmospheres.

Tropospheric sources of wave disturbances in the zonal flow are baroclinic instability and the forcing action of zonally asymmetric heating and topography. The transmissivity of the upper atmosphere increases with wavelength and is greater for the forced perturbations than for the unstable tropospheric waves, whose lengths must be smaller than the critical length for instability. The analysis indicates that baroclinically unstable wave disturbances originating in the troposphere probably do not propagate energy vertically at all.

When energy is propagated to great heights, nonlinear vertical eddy transports of heat and momentum associated with the vertically propagating waves should modify the basic zonal flow. However, when the wave disturbance is a small stationary perturbation on a zonal flow that varies vertically but not horizontally, the second-order effect of the eddies on the zonal flow is zero.

1. INTRODUCTION

two types of energy sources

Motions in the upper atmosphere are of two kinds, those whose immediate sources of energy are in the upper atmosphere itself and those whose energy is transmitted from the lower atmosphere. An example of the former is ionospheric turbulence. An example of the latter is the solar semidiurnal tide, in which the gravitational attraction of the sun, acting on the lower, massive, part of the atmosphere, produces an upward-propagating gravity wave. Certain irregular motions in the D and lower E regions of the ionosphere, which have been revealed by observations of meteor trails, may also, as has been suggested by *Hines* [1959], be due to the propagation of gravity waves from the lower atmosphere.

Little is known of the long-period, planetary-scale motions in the upper atmosphere, although motions of this type in the troposphere contain the bulk of the atmosphere's energy. To what extent are such disturbances coupled to the motions in the troposphere? To what extent is the so-called breakdown of the polar-night jet in the stratosphere associated with motions in the troposphere? (The causes of the breakdown are discussed by *Murray* [1960].) How much of the energy in the troposphere propagates into the upper atmosphere? The answers to these questions would seem to be central to an understanding of the planetary-scale motions of the upper atmosphere. Thus it has long been a source of wonder to one of us that the upper-air motions are not coupled in a more obvious manner to those in the lower atmosphere, as, for example, the motions in the solar chromosphere and corona are thought to be coupled to those in the convective layer of the sun [cf. *Kuiper*, 1953]. The tidal oscillations, as well as the gravity waves studied by Hines, travel upward with a slowly decreasing kinetic energy density. If the large-scale tropospheric motions were to propagate in this manner, then, because of their

[1] Present address: Department of Mathematics, Bristol University, Bristol, England.

vastly greater energy, an atmospheric corona would in all likelihood be produced. The kinetic energy density in the lower troposphere is of the order of 10^3 ergs cm^{-3}. If this energy were to travel upward with little attenuation and be converted into heat by friction or some other means at, say, 100 km, where the density is diminished by a factor of 10^{-6}, it would raise the air temperature to about 100,000°K. At such temperatures most of the atmosphere would escape the earth's gravitational field. It is important for the understanding of the upper atmospheric motions to know why this does not occur, i.e., why the tropospheric energy is so effectively trapped.

trapping of troposphere energy

The reverse problem, the mechanical propagation of energy from the upper into the lower atmosphere, has received some attention because of hopes of associating weather changes with fluctuations in electromagnetic or corpuscular radiation from the sun. At the moment it seems that these hopes are doomed to disappointment, for by any physical consideration so far advanced the energies involved are insignificant.

Vertical propagation in planetary wave systems has previously been studied by *Charney* [1949], who found the speed of energy propagation in the vertical (vertical component of group velocity) to have a maximum value of about 5 km per day at middle latitudes. This work was extended by *Ooyama* [1958] to the downward propagation of a pulse in a resting atmosphere. His results 'lent little support to proposals of the existence of anomalous solar-weather relationships.'

Charney was concerned with the upper atmosphere only as a boundary for the lower atmosphere, and both he and Ooyama took the atmosphere to be at rest with respect to the moving earth. But it can be shown that the transmissivity of the upper atmosphere to planetary waves is exceedingly sensitive to the mean zonal wind structure. Indeed, we shall show in the present paper that it is primarily the variation of mean zonal wind with height that gives rise to the energy trapping. In the following, we present the results of an investigation into the vertical propagation of planetary-wave disturbances in an atmosphere with arbitrary vertical gradients of temperature and mean zonal wind.

2. Derivation of the Wave Equation

We shall be concerned with wave disturbances whose horizontal wavelengths are comparable in size with the earth's radius and whose orbital periods are large compared with the period of the earth's rotation. Such disturbances are certainly quasi-hydrostatic in the sense that the vertical components of the forces of pressure and gravity are nearly in balance. That they are also quasi-geostrophic, i.e., that the horizontal components of the pressure and Coriolis forces are nearly in balance, may be seen as follows. Let U be a characteristic horizontal particle or phase velocity, S a characteristic horizontal length, and Ω the angular speed of the earth's rotation. We construct the Rossby number

$$R_0 \equiv U/\Omega S$$

quasi geostroph

which measures the ratio of the horizontal component of the inertial force to that of the Coriolis force. Since this number is of the order 10^{-1} or less for the planetary waves, the flow is quasi-geostrophic.

The equations of motion in the small Rossby number regime may be greatly simplified by the systematic use of the hydrostatic and geostrophic approximations to 'filter out' the irrelevant high-frequency motions [*Charney*, 1948]. Let p be the pressure, ρ the density, $\mathbf{v}$ the velocity, g the acceleration of gravity, z the upward-directed vertical coordinate, $\mathbf{k}$ a unit vector in the direction of z, and ϕ the latitude; then the hydrostatic and geostrophic equations are

$$-g - \rho^{-1}\,\partial p/\partial z = 0 \qquad (2.1)$$

hydro

$$f\mathbf{v}_h \times \mathbf{k} - \rho^{-1}\,\mathrm{grad}_h\, p = 0 \qquad (2.2)$$

geostroj

where $f = 2\Omega \sin\phi$, and the subscript h is used to denote the horizontal component of a vector.

We assume that the atmosphere is a perfect gas. Its specific entropy

$$\begin{aligned} s &= c_p \ln\theta + \text{constant} \\ &= c_v \ln p - c_p \ln\rho + \text{constant} \end{aligned} \qquad (2.3)$$

entropy

where θ is the potential temperature, and c_v and c_p are the specific heats at constant volume and constant pressure, respectively. For adiabatic motion

$$D \ln\theta/Dt + w\,\partial \ln\theta/\partial z = 0 \qquad (2.4)$$

horizontal this is total derivative = constant.

where w is the vertical velocity component and D/Dt is the operator $\partial/\partial t + \mathbf{v}_h \cdot \text{grad}$.

For small Rossby numbers, the equation for the vertical component ζ of the relative vorticity, curl $\mathbf{v}$,

vorticity equation

$$D(\zeta + f)/Dt + (\zeta + f) \text{ div } \mathbf{v}_h$$
$$= -\text{div } [(w\, \partial \mathbf{v}_h/\partial z + \rho^{-1} \text{ grad}_h\, p) \times \mathbf{k}]$$

simplifies to

not geostrophic.

quasi-geostrophic

$$D(\zeta + f)/Dt + f \text{ div } \mathbf{v}_h = 0 \qquad (2.5)$$

it being understood that ζ and the $\mathbf{v}_h$ appearing in D/Dt, but not in the divergence term, are to be evaluated from the geostrophic equation. In the derivation of the above equation it is tacitly assumed that the Richardson number

$$Ri \equiv g\, D^2[\partial \ln \bar{\theta}/\partial z]\, U^{-2} \gg 1$$

where $\bar{\theta}$ is a horizontal mean of θ and D is either the vertical scale height or the characteristic vertical scale of the disturbance. Actually the Richardson number is of the order 10^2, so that $(RoRi)^{-1}$ is also small. If, in addition, D is small in comparison with the vertical scale of variation of θ, it may be shown that the horizontal variations of p, ρ, and θ are small compared with their respective mean values, so that ρ in equation 2.2 may be replaced by $\bar{\rho}$, a function of z alone, and that the hydrostatic equation may be replaced by

$$g\, \delta(\ln \theta) \cong (\partial/\partial z)(\delta p/\bar{\rho}) \qquad (2.6)$$

where the δ denotes a deviation from a horizontal mean value, i.e., $\delta p \equiv p - \bar{p}(z)$ and $\delta(\ln \theta) \equiv \ln \theta - \ln \bar{\theta}(z)$.

A second tacit assumption is involved in the derivation of equation 2.5. It has been pointed out by *Burger* [1958] that the horizontal scale of the disturbance must be small compared with the scale of variation of the Coriolis parameter f. This might make it appear that disturbances on a truly planetary scale are precluded. It is found, however, that the regions to which the disturbances are confined are sufficiently limited laterally to make the assumption valid.

β-plane

With the definition $\chi \equiv \delta p/\bar{\rho} f$, the foregoing approximations permit the hydrostatic and geostrophic equations, 2.1 and 2.2, to be written

hydro.

$$g\, \delta(\ln \theta)/f = \partial \chi/\partial z \qquad (2.7)$$

geostrophic

$$\mathbf{v}_h = \mathbf{k} \times \text{grad}_h\, \chi \qquad (2.8)$$

the latter giving

$$\zeta = \text{div grad}_h\, \chi \qquad (2.9)$$

The approximations also lead to the simplified forms

thermo equation

$$\frac{D}{Dt}\left(\frac{\partial \chi}{\partial z}\right) + \frac{gw}{f}\frac{\partial \ln \bar{\theta}(z)}{\partial z} = 0 \qquad (2.10)$$

and

continuity equat

$$\bar{\rho} \text{ div } \mathbf{v}_h + \partial(\bar{\rho} w)/\partial z = 0 \qquad (2.11)$$

for the adiabatic and continuity equations, respectively. Elimination of div $\mathbf{v}_h$ between equations 2.5 and 2.11 gives

$$D(\zeta + f)/Dt = f\, \partial(\bar{\rho} w)/\bar{\rho}\, \partial z \qquad (2.12)$$

which, together with the adiabatic equation 2.10, completely determines the motion if the variables $\mathbf{v}_h$ and ζ are evaluated geostrophically by means of equations 2.8 and 2.9.

For simplicity we refer the motion to the Rossby β plane, a device that enables us to ignore the unimportant but complicated geometrical effect of the earth's curvature while retaining its dynamical effect. The curvature of the earth is ignored, but the variability of f is retained on the right-hand side of equation 2.12; elsewhere it is set equal to a mean value f_0 corresponding to a mean latitude ϕ_0 which is usually taken to be 45°. We take a Cartesian coordinate system with the x axis directed eastward and the y axis northward. The velocity components in these directions are denoted by u and v.

β plane

?

In the undisturbed state of the atmosphere the potential temperature is assumed to vary vertically and horizontally according to the law

ln - linear in y

$$\ln \theta_0(y, z) = \ln \bar{\theta}(z) + y A(z) \qquad (2.13)$$

where the subscript 0 denotes a quantity in the undisturbed state. Differentiating (2.7) with respect to y and substituting from (2.8), we obtain

$$\partial u_0/\partial z = -g A(z)/f_0$$

Hence if u_0 is constant at one level, as we shall assume, it is a function of height only.

Denoting perturbation quantities by primes,

$\chi = \dfrac{\delta p}{\bar{\rho} f}$ $\qquad$ δp = deviation from average pressure.

we may write for the perturbation forms of equations 2.12 and 2.10

perturbation
$$\left(\frac{\partial}{\partial t} + u_0 \frac{\partial}{\partial x}\right)\nabla_h^2 \chi' + \beta \frac{\partial \chi'}{\partial x} = f_0\left(\frac{\partial}{\partial z} - \frac{1}{H}\right)w' \qquad (2.14)$$

and

$$\left(\frac{\partial}{\partial t} + u_0 \frac{\partial}{\partial x}\right)\frac{\partial \chi'}{\partial z} - \frac{\partial u_0}{\partial z}\frac{\partial \chi'}{\partial x} + \frac{N^2}{f_0} w' = 0 \qquad (2.15)$$

where ∇_h is the horizontal grad operator, H is the scale height, $-[\partial(\ln \bar{p})/\partial z]^{-1}$, and N is the Brunt-Vaisälä frequency for the undisturbed motion

$$N^2 \equiv g \ln \bar{\theta}/\partial z \qquad (2.16)$$

The quantity $\beta \equiv df/dy$ is assumed to have an appropriate constant value corresponding to the latitude for which $f = f_0$. The important dynamical effect on the planetary motions of the variation of temperature with height is determined by N, which may be interpreted physically as the frequency of a vertical buoyancy oscillation. The scale height H is given by

scale height
$$\frac{1}{H} = \frac{g}{RT} + \frac{1}{T}\frac{\partial T}{\partial z} \cong \frac{g}{RT} \qquad (2.17)$$

where R is the gas constant referred to unit mass of air and T is the absolute temperature. Since H in equation 2.14 only measures the inertial effect of the variation of density with height, it is permissible to replace it, where convenient, by a mean value corresponding to a characteristic average temperature T; it is well known that the exponential law of decay

$$\bar{p}(z) = \bar{p}(0)e^{-z/H}$$

approximates the observed distribution of density well in the first 100 km.

Since the coefficients of the terms in the perturbation equations are functions of z alone, we seek perturbations with independent wave components of the form of a product of some function of z and exp $i(kx + ly - kct)$. Let us use capital letters to denote the functions of z; thus

$$\chi' = X(z)e^{i(kx+ly-kct)}$$

$$v' = V(z)e^{i(kx+ly-kct)} \quad \text{etc.}$$

Substituting these into equations 2.14 and 2.15, and then eliminating $\frac{\partial \chi'}{\partial x} \propto V$

$$W = -\frac{f_0}{N^2}\left[(u_0 - c)\frac{dV}{dz} - \frac{du_0}{dz}V\right] \qquad (2.18)$$

we get

$$f_0^2\left(\frac{d}{dz} - \frac{1}{H}\right)\left\{\frac{1}{N^2}\left[(u_0 - c)\frac{dV}{dz} - \frac{du_0}{dz}V\right]\right\} - (k^2 + l^2)(u_0 - c - u_c)V = 0 \qquad (2.19)$$

where WAVE Equation

$$u_c \equiv \beta/(k^2 + l^2) \qquad (2.20)$$

is the Rossby critical velocity. When H, N^2, and du_0/dz are constants, equation 2.19 reduces to an equation found by *Charney* [1947]. Our derivation of this equation for V from the a priori quasi-geostrophic equations is essentially equivalent to his derivation from the general equations of motion by the systematic use of small Rossby and large Richardson number approximations.

Wave equation in spherical polar coordinates. An alternative treatment can illustrate the nature of the β-plane approximation. If z is the radial coordinate, ϕ the latitude, λ the longitude, and a the radius of the earth, the wave equation in spherical polar coordinates can be found as follows. The zonal velocity $u = -\partial\chi/a\partial\phi$ and the meridional velocity $v = \partial\chi/a \cos\phi\, \partial\lambda$ now, if $z - a \ll a$. With

$$\ln \theta_0 = \ln \bar{\theta}(z) + B(z)\sin^2\phi \qquad (2.13')$$

we get

$$u_0 = a\omega_0(z)\cos\phi$$

where $d\omega_0/dz = -gB/2\Omega a^2$ in order to satisfy the mean hydrostatic and geostrophic equations. Elimination of w' from equations 2.14 and 2.15 expressed in spherical polars gives

$$\left(\frac{\partial}{\partial t} + \omega_0\frac{\partial}{\partial\lambda}\right)\nabla_h^2\chi' + \frac{2\Omega}{a^2}\frac{\partial\chi'}{\partial\lambda} = -f^2\left(\frac{\partial}{\partial z} - \frac{1}{H}\right)\left\{\frac{1}{N^2}\left[\left(\frac{\partial}{\partial t} + \omega_0\frac{\partial}{\partial\lambda}\right)\frac{\partial\chi'}{\partial z} - \frac{d\omega_0}{dz}\frac{\partial\chi'}{\partial\lambda}\right]\right\}$$

without the need to assume that f or β is constant. However, we must now make the assumption that f is constant on the right-hand side of (2.14) and in (2.15) in order to separate the variables with

$$\chi' = X(z)e^{i(m\lambda+\sigma t)}P_n^m(\sin\phi)$$

where P_n^m is the associated Legendre polynomial

$$P_n^m(\sin\phi) \equiv \cos^m\phi\, d^m P_n(\sin\phi)/d(\sin\phi)^m$$

P_n being the ordinary Legendre polynomial. Then

$$f_0^2\left\{\left(\frac{d}{dz}-\frac{1}{H}\right)\frac{1}{N^2}\left[\left(\omega_0+\frac{\sigma}{m}\right)\frac{dX}{dz}-\frac{d\omega_0}{dz}X\right]\right\} - \frac{n(n+1)}{a^2}\left[\omega_0+\frac{\sigma}{m}-\frac{2\Omega}{n(n+1)}\right]X = 0 \qquad (2.19')$$

The analogy with equation 2.19 is apparent. In spite of the geometrical difference of the coordinate frames, the equations are in fact identical if we set $\beta = 2\Omega\cos\phi_0/a$, $u_0 = a\omega_0\cos\phi_0$, $k^2 + l^2 = n(n+1)/a^2$, and $-c = a\sigma\cos\phi_0/m$, for the mean latitude ϕ_0 at which $f = f_0$.

Energy equation. We conclude this section by deriving the equation for the time rate of change of the perturbation energy. Define a typical perturbation α' such that

define

$$\alpha = \alpha_0(y, z, t) + \alpha'(x, y, z, t)$$

where

$$\int \alpha'\, dx = 0$$

the integral extending over a period of the motion. If equations 2.12 and 2.10 are averaged with respect to x and subtracted from the unaveraged equations, one obtains

$$\bar\rho\nabla_h^2\,\partial\chi'/\partial t = f\,\partial(\bar\rho w')/\partial z - \bar\rho[\mathbf{v}\cdot\nabla_h(\zeta+f)]'$$

$$\frac{f_0^2\bar\rho}{N^2}\frac{\partial^2\chi'}{\partial t\,\partial z} = -f\bar\rho w' - \frac{f^2\bar\rho}{N^2}\left[\mathbf{v}\cdot\nabla_h\frac{\partial\chi}{\partial z}\right]'$$

MANIP. Multiply the first by $-\chi'$, the second by $\partial\chi'/\partial z$; add, and integrate over the atmosphere between the levels z_1 and z_2. After some manipulation, integration by parts, and use of the geostrophic relationship (2.8), the perturbation energy equation is obtained:

$$\frac{d}{dt}\int_\tau\left[\frac{1}{2}\bar\rho(\nabla_h\chi')^2 + \frac{1}{2}\bar\rho\frac{f^2}{N^2}\left(\frac{\partial\chi'}{\partial z}\right)^2\right]d\tau = \int_\tau\bar\rho\frac{\partial\chi'}{\partial x}\frac{\partial\chi'}{\partial y}\frac{\partial u_0}{\partial y}\,d\tau + \int_\tau\bar\rho\frac{f^2}{N^2}\frac{\partial\chi'}{\partial x}\frac{\partial\chi'}{\partial z}\frac{\partial u_0}{\partial z}\,d\tau \quad \text{energy equation}$$

$$\left[\int_S\bar\rho f\chi' w'\,dS\right]_{z=z_1}^{z=z_2} \qquad (2.21)$$

where τ denotes a volume and S a horizontal surface. The first term in the integrand on the left-hand side is the perturbation kinetic energy, and the second, the perturbation 'available' [*Lorenz*, 1955] potential energy. The first integral on the right is the rate of conversion of mean flow kinetic energy into perturbation kinetic energy; the second is the rate of conversion of mean flow potential energy into perturbation potential energy; and the last is the rate of flow of perturbation energy into the volume. From the definition of χ this last term may be written in the more familiar form

Boundary term

$$-\left[\int_S p'w'\,dS\right]_{z=z_1}^{z=z_2} \qquad (2.22)$$

We note that the kinetic energy of the vertical motion and the vertical eddy stresses are absent. This is because the vertical velocities are extremely small in long-period planetary flows.

3. Solution of the Wave Equation

Use of analogies. The differential equation 2.19

$$(u_0 - c)\frac{d}{dz}\left(\frac{\bar\rho}{N^2}\frac{dV}{dz}\right) - \left[\frac{d}{dz}\left(\frac{\bar\rho}{N^2}\frac{du_0}{dz}\right) + \frac{\beta\bar\rho}{f_0^2u_c}(u_0 - c - u_c)\right]V = 0$$

for baroclinic waves in an inviscid atmosphere can be transformed into the canonical form

$$d^2\Xi/dz^2 + n^2\Xi = 0 \qquad (3.1)$$

if

$$\Xi \equiv (\bar\rho/N^2)^{1/2}V$$

and

$$n^2 \equiv -\left\{\frac{(k^2+l^2)N^2}{f_0^2} + \sqrt{\frac{N^2}{\bar{p}}}\frac{d^2}{dz^2}\sqrt{\frac{\bar{p}}{N^2}}\right\} + \frac{N^2}{u_0 - c}\left\{\frac{\beta}{f_0^2} - \frac{1}{\bar{p}}\frac{d}{dz}\left(\frac{\bar{p}}{N^2}\frac{du_0}{dz}\right)\right\} \quad (3.2)$$

We shall often find it convenient to make this equation dimensionless. Thus we define $\nu \equiv 2H_0 n$, $\zeta \equiv z/2H_0$ so that

$$d^2\Xi/d\zeta^2 + \nu^2\Xi = 0 \quad (3.3)$$

where H_0 is some (constant) characteristic scale height. There is no solution of the equation in terms of known functions for general $\nu(\zeta)$. However, analogies with the electromagnetic and Schrödinger wave equations are useful. The above equation is like that of one-dimensional wave propagation in a medium of variable refractive index $\nu(\zeta)$, and of one-dimensional transmission of particles by wave mechanics. Further, the boundary conditions we shall derive determine the direction of the energy flux, as in the theories of electromagnetic waves and particle beams. So the analogies indicate the qualitative nature of our solution (we shall confirm these indications later). In regions where the 'refractive index' ν is pure imaginary there are external waves (i.e., V varies exponentially with z) and vertical propagation of energy is inhibited, there being some trapping (i.e., reflection) of waves. An infinite layer in which ν is imaginary will reflect waves totally, so that the net vertical propagation of energy is zero in a steady state. In regions where ν is real there are internal waves (i.e., V is oscillatory in z) and vertical propagation of energy is freely permitted. If ν^2 is positive near the ground and high up, but negative in an intermediate layer, there will be partial reflection in the middle layer, as occurs in the tunnel effect of wave mechanics.

ν imag. "external waves"

ν real internal waves

Solution for constant velocity and temperature. More information comes from the exact solution of the wave equation in the simplest case, that of constant mean velocity and temperature. In this case $\partial(\ln \bar{p})\,\partial z = -1/H$ and $N^2 = g(\gamma - 1)/\gamma H$ are constants, and so the dimensional equation for V becomes

$$\frac{d^2V}{dz^2} - \underbrace{\frac{1}{H}\frac{dV}{dz}} - \frac{\beta N^2(u_0 - c - u_c)}{f_0^2 u_c(u_0 - c)}V = 0 \quad (3.4)$$

this is expected to have some sort of damping effect

γ being the ratio c_p/c_v of the specific heats. Therefore

$$V = (Ae^{inz} + Be^{-inz})e^{z/2H}$$

for some complex constants A, B, where

$$n^2 = -\tfrac{1}{4}H^{-2} - N^2 f_0^{-2} \cdot \{(k^2 + l^2) - \beta/(u_0 - c)\} \quad (3.5)$$

with $Re\, n > 0$, or $Re\, n = 0$ and $Im\, n > 0$, for definiteness. The condition $n^2 > 0$ for internal waves when c is real is

$$0 < u_0 - c < \beta/\{(k^2 + l^2) + f_0^2/4H^2N^2\} \equiv U_c \quad (3.6)$$

internal and external

say. If $u_0 - c < 0$, or $u_0 - c > U_c$, the waves are external.

It can be seen that the temperature affects U_c through $H^2N^2 = g(\gamma - 1)H/\gamma$ only. Now the variation of the scale height is quite small ($\lesssim$ 25 per cent) in the atmosphere's lowest 100 km, and so U_c is nearly independent of the temperature and depends principally on the wave numbers.

In the quasi-geostrophic approximation the horizontal mean of $\frac{1}{2}\bar{p}(u'^2 + v'^2)$ is the same as that of $\frac{1}{2}\bar{p}(1 + k^2/l^2)v'^2$. Therefore the horizontal mean of the kinetic energy density in the case of constant mean velocity and temperature, ?

K.E. density

$$Q_k = \tfrac{1}{4}\bar{p}(1 + k^2/l^2)e^{z/H}\{|A|^2 e^{i(n-n^*)z} + AB^*e^{i(n+n^*)z} + A^*Be^{-i(n+n^*)z} + |B|^2 e^{i(n^*-n)z}\}e^{2kc_it}$$

where asterisks denote complex conjugates. Also the horizontal mean of the vertical energy flux

$$\begin{aligned} W_z &\equiv \overline{p'w'} \\ &= Re(f_0\bar{p}v'/ik)Re(-f_0N^2(u_0 - c)\,\partial v'/\partial z) \\ &= \tfrac{1}{4}f_0^2k^{-1}N^{-2}u_0\bar{p}e^{z/H}\{(n+n^*)(|A|^2e^{i(n-n^*)z} \\ &\quad - |B|^2e^{i(n^*-n)z}) + (n - n^*)(A^*Be^{i(n+n^*)z} \\ &\quad - A^*Be^{-i(n+n^*)z})\}e^{2kc_it} \\ &= \begin{cases} \tfrac{1}{2}f_0^2k^{-1}N^{-2}u_0n\bar{p}e^{z/H} \cdot (|A|^2 - |B|^2)e^{2kc_it} & (n \text{ real}) \\ \tfrac{1}{2}f_0^2k^{-1}N^{-2}u_0n\bar{p}e^{z/H} \cdot (AB^* - A^*B)e^{2kc_it} & (n \text{ imaginary}) \end{cases} \end{aligned}$$

The meridional velocity perturbation v' behaves like $\bar{\rho}^{-1/2}e^{\pm inz}$. Therefore W_z is independent of height, and Q_k varies like

$$\{|A|^2 + AB^*e^{2inz} + A^*Be^{-2inz} + |B|^2\}$$

for internal waves (real n), and like

$$|A|^2 e^{2inz} + AB^* + A^*B + |B|^2 e^{-2inz}$$

for external waves (imaginary n). We shall find that the damped solution ($V \propto e^{(1/2H+in)z}$) is preferred when the energy source is below (cf. the tunnel effect of wave mechanics). Then the energy flow through a reflecting layer (i.e., a region with imaginary n) of thickness h is of the order of $e^{-|2hn|}$ times the incident energy flux. The values of $\nu = 2Hn$ in the atmosphere can be seen from Figures 5 and 8, and give an idea of the rate of trapping.

Other exact solutions. (i) When

$$u_0 = \text{constant} \qquad H = H_0 + \kappa z$$

we find

$$\bar{\rho} = \rho_0(z + H_0/\kappa)^{-(1+\kappa^{-1})}$$

$$N^2 = \frac{g}{H}\left(\frac{\gamma - 1}{\gamma} + \kappa\right)$$

Therefore

$$\frac{d^2V}{dz^2} - \frac{1}{\kappa(z + H_0/\kappa)}\frac{dV}{dz} + \frac{b}{z + H_0/\kappa}V = 0$$

where

$$b \equiv -\frac{g}{f_0^2}\left(1 + \frac{\gamma - 1}{\gamma\kappa}\right) \cdot [(k^2 + l^2) - \beta/(u_0 - c)]$$

Therefore

$$V = (z + H_0/\kappa)^{\frac{1}{2}(1+\kappa^{-1})} \cdot Z_{1+\kappa^{-1}}(2[b(z + H_0/\kappa)]^{1/2})$$

where $Z_{1+\kappa^{-1}}$ is a Bessel function of order $(1 + \kappa^{-1})$.

The asymptotic solution

$$V \sim (z + H_0/\kappa)^{(1+2\kappa^{-1})/4} \cdot \exp\{2[-b(z + H_0/\kappa)]^{1/2} - \pi(\tfrac{1}{4} + \tfrac{1}{2}\kappa^{-1})\}$$

The waves are internal if $b > 0$, i.e. if $0 < u_0 - c < u_c$. We find (using the condition of bounded kinetic energy density Q_k, and assuming that c is real) that, as $z \to \infty$,

$$Q_k \sim \begin{cases} \text{constant} \times z^{-1/2} \\ z^{1/2}e^{-4(-bz)^{1/2}} \end{cases}$$

$$W_z \to \begin{cases} \text{constant} & \text{(internal waves)} \\ 0 & \text{(external waves)} \end{cases}$$

(ii) When

$$u_0 = U_0 + \Lambda z \qquad H = \text{constant}$$

$$\frac{d^2V}{dz^2} - \frac{1}{H}\frac{dV}{dz} - a^2\frac{u_0 - c - \bar{u}_c}{u_0 - c}V = 0$$

where

$$a^2 \equiv g(k^2 + l^2)(\gamma - 1)/\gamma f_0^2 H$$

and

$$\bar{u}_c \equiv u_c + \Lambda/a^2H$$

It has been shown [*Charney*, 1947] that the solution

$$V = \psi \exp[(\tfrac{1}{2}H^{-1} - \bar{a})z]$$

if $\psi(\xi; r)$ satisfies the confluent hypergeometric equation

$$\xi\psi_{\xi\xi} - \xi\psi_\xi + r\psi = 0$$

where

$$\xi \equiv (2\bar{a}/\Lambda)(u_0 - c)$$

$$\bar{a}^2 \equiv a^2 + \tfrac{1}{4}H^{-2} \qquad r \equiv a^2\bar{u}_c/2\bar{a}\Lambda$$

Two independent solutions are

$$\psi_1(\xi; r) = \begin{cases} 1 - r\xi M(1 - r, 2, \xi)[\ln\xi \\ \quad + \Gamma'(-r)/\Gamma(r) - 2\Gamma'(1)/\Gamma(1) \\ \quad + \sum_{n=1}^{\infty}\sum_{\nu=0}^{n-1}\left[\left(\frac{1}{\nu - r} - \frac{2}{\nu + 1}\right) + \frac{1}{n}\right] \\ \quad \cdot \frac{-r(1 - r)\cdots(-r + n - 1)}{(n - 1)!\,n!}\xi^n & (r \neq 1) \\ 1 - \xi\left[\ln\xi + \sum_{n=1}^{\infty}\frac{\xi^n}{(n + 1)!\,n}\right] & (r = 1) \end{cases}$$

$$\psi_2(\xi, r) \equiv -r\xi M(1 - r, 2, \xi)$$

where

$$M(1 - r, 2, \xi) = 1 + \frac{(1-r)}{1!\,2!}\xi + \frac{(1-r)(2-r)}{2!\,3!}\xi^2 + \cdots *$$

The asymptotic expansions of ψ_1 and ψ_2 are

$$\psi_1 \sim [(-\xi)^r/\Gamma(r)]G(-r, 1 - r; -\xi)$$

$$\psi_2 \sim [\xi^{-r}e^{\xi}/\Gamma(-r)]G(1 + r, r; \xi)$$

where

$$G(\mu, \nu; \xi) \equiv 1 + \frac{\mu\nu}{1!\,\xi} + \frac{\mu(\mu+1)\nu(\nu+1)}{2!\,\xi^2} + \cdots$$

Hence $Q_k \sim \exp\{-(4a^2 + H^{-2})^{1/2}z\}$, and the waves are external at infinity where u_0 is infinite. However, vertical energy propagation below the branch point $\xi = 0$ of the logarithm is possible (cf. an example in section 7).

Some asymptotic solutions. We can gain a little more insight from the asymptotic solution of the wave equation. A few cases of asymptotic solutions in terms of known functions follow.

(i) When

$$u_0 = \text{constant} \qquad H = H_0 e^{\kappa z/H_0} \quad (\kappa > 0)$$

we find $\bar{\rho} \sim \rho_0 e^{-\kappa z/H_0}$, $N^2 \to g\kappa/H_0$ as $z \to \infty$. Therefore

$$V \sim e^{(1/2\kappa/H_0 \pm in)z}$$

where

$$n^2 = -\kappa^2/4H_0^2 - g\kappa f_0^{-2}H_0^{-1} \cdot \{(k^2 + l^2) - \beta/(u_0 - c)\}$$

When c is real the waves are internal if

$$0 < u_0 - c < \beta/\{(k^2 + l^2) + f_0^2 H_0/4g\kappa\}$$

(ii) $u_0 =$ constant, $H = H_0\, e^{-\kappa z/H_0}$ $(\kappa > 0)$. Here

$$\bar{\rho} = \rho_0 \exp(\kappa z/H_0 - H_0/\kappa H)$$

$$N^2 \sim g(\gamma - 1)/\gamma H$$

* This ψ_1 is the same as Charney's ψ_1 multiplied by $\pi/\sin(-r\pi)$. However, through an error, the function tabulated in his table 1 is actually the present ψ_1.

Therefore

$$V \sim e^{-b_1 z} \quad \text{or} \quad e^{(b_1 - \kappa/H_0)z + H_0/\kappa H} \quad \text{as} \quad z \to \infty$$

where

$$b_1 \equiv \frac{g(\gamma - 1)}{f_0^2\gamma}\{(k^2 + l^2) - \beta/(u_0 - c)\}$$

the waves being always external for real c. As $z \to \infty$, $Q_k \sim e^{-H_0/\kappa H}$.

(iii) $u_0 = U_0 + \Lambda z$, $H = H_0 + \kappa z$ $(\kappa > 0)$. Here

$$V \sim z^{1/4(1+2K^{-1})} \exp\{\pm 2(b_2 z/\kappa)^{1/2}\} \quad \text{as} \quad z \to \infty$$

where

$$b_2 \equiv g\beta\left(\frac{\gamma - 1}{\gamma} + \kappa\right)/f_0^2 u_c$$

If c is real the waves are always external and

$$Q_k \sim z^{1/2} \exp\{-4(b_2 z/\kappa)^{1/2}\} \quad \text{as} \quad z \to \infty$$

(iv) $u_0 = U_0 + \Lambda z$, $H = H_0 e^{\kappa z H_0} (\kappa > 0)$. Here

$$V \propto e^{(1/2\kappa/H_0 \pm in)z}$$

where

$$n^2 \equiv -\kappa^2/4H_0^2 - f_0^{-2}g\kappa\, H_0^{-1}(k^2 + l^2)$$

as in case ii. All waves are external.

(v) $u_0 = U_0 + \Lambda z$, $H = H_0 e^{-\kappa z/H_0}$ $(\kappa > 0)$. As in case ii,

$$V \sim e^{-b_3 z}$$

$$\text{or} \quad \exp\{(a^2 - \kappa/H_0)z + H_0/\kappa H\}$$

where now

$$b_3 \equiv g(\gamma - 1)(k^2 + l^2)/f_0^2\gamma$$

All waves are external.

General solution. For cases where there is no exact solution we must use direct numerical integration or some method of approximation. Because ν changes considerably in a scale height of the real atmosphere, the W.K.B. and like methods do not seem suitable (cf. *Eckart* [1960] for a discussion on the use of these methods for gravity waves in the atmosphere). It seems best to use solutions in layers in which u_0, H are constant or vary linearly with height. The boundary conditions of continuous pressure and vertical velocity enable solutions in adjacent layers to be joined up. The advantages, and limitations, of this method are illustrated by *Eckart's* [1930] exact solution of Schrödinger's equation for a class of smoothly varying functions which approximate the potential well.

4. Methods of Excitation of Planetary Waves

Planetary wave perturbations may be produced by the action of an external force or may be generated spontaneously by some form of hydrodynamic instability. Forced perturbations are produced mechanically by the action of continental elevations on the undisturbed zonal flow [*Charney and Eliassen*, 1949] and thermally by the action of differential heating over the continents and oceans [*Smagorinsky*, 1953]. Self-excited perturbations are due primarily to baroclinic instability [*Charney*, 1947; *Eady*, 1949]. Let us first consider the forced perturbations. As we are concerned only with the general nature of the vertical propagation of disturbances and not with their detailed structure, we may assume that the surface topography and the distribution of the sources and sinks of heat have a sinusoidal variation. The influence of topography may be expressed simply as a boundary condition on the vertical velocity at the mean elevation of the ground ($z = 0$):

$$w'(x, y, 0) = u_0(0)\, \partial h/\partial x \qquad (4.1)$$

where we assume the height of the earth's surface

$$h = h_0 e^{ikx} \cos ly \qquad (4.2)$$

From the data given by *Charney and Eliassen* [1949] we estimate $h_0 = 1$ km, $u_0(0) = 5$ m sec^{-1}, and $2\pi/k = 14{,}000$ km, corresponding to the azimuthal wave number $m = ka \cos \phi_0 = 2$ at $\phi_0 = 45°$, i.e. to two continents and two oceans. From these values we obtain 0.2 cm sec^{-1} for the amplitude of w'. It is assumed that h is small in comparison with the vertical scale of the disturbance and that $u_0(0)$ is not too small. A critical appraisal of these assumptions is given in the Appendix.

Thermal effects may also be represented by a condition on the surface vertical velocity if it is assumed that the heating and cooling occur in a relatively thin layer near the surface and that the horizontal heat transport in this layer is negligible. The first assumption is founded on the facts that differential heating takes place mainly by condensation, evaporation, and turbulent conduction and that the horizontal gradient of heating by infrared and solar radiation is small. Under these assumptions the flux of heat, F, into the heated layer must be removed from the top of the layer by large-scale convection. Hence if w_1' is the vertical velocity component in the planetary wave, $\bar{\rho}_1$ the mean density, and $\bar{T}_1$ the mean temperature at the top of the layer,

$$\bar{\rho}_1 w_1' c_p \bar{T}_1 = F$$

Smagorinsky's [1953] estimate of 0.3 cal cm^{-2} min^{-1} for the amplitude of F gives 0.1 cm sec^{-1} for the amplitude of w_1'. However, this estimate is at best an upper bound on the magnitude of w_1', for one cannot justify the neglect of horizontal heat transfer within the heated layer. It has been shown merely that the perturbing effect of differential heating is at most comparable in magnitude to that of topography.

At temperate and subtropical latitudes in the northern hemisphere the bulk of the tropospheric kinetic energy is distributed about equally between the low azimuthal wave numbers 1, 2, 3 and the middle azimuthal wave numbers 5, 6, 7, 8. At high latitudes the distribution is skewed more toward the low wave numbers [*Saltzman*, 1958]. The disturbances in the former category are mechanically or thermally forced. Those in the latter are a consequence of the baroclinic instability of the strong middle-latitude westerlies. It is unlikely, however, that energy in the latter category is propagated to very high altitudes, because the critical wavelength beyond which the waves are stable is too small to permit vertical transmission, except in circumstances where the mean zonal velocities at high levels are very small and positive.

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To see this let us examine the case where the mean zonal velocity increases linearly with height up to the tropopause ($z = h_1$) and remains constant thereafter, N^2 being assumed constant in each layer. It was shown by *Charney* [1947] that the stability criterion is then nearly the same as that for an atmosphere in which the zonal velocity continues to increase to infinity. One may therefore use the criterion for the latter case, which is expressed concisely by the inequality [*Gambo*, 1950; *Kuo*, 1952]

$$r \equiv (\beta N^2/f_0^2\Lambda + 1/H) \cdot \{1/H^2 + 4f_0^{-2}N^2(k^2 + l^2)\}^{-1/2} < 1$$

or

$$k^2 + l^2 > \beta/2\Lambda H + \beta^2 N^2/4f_0^2\Lambda^2$$

where $\Lambda = du_0/dz$. It was shown in section 3 that the criterion for transmission ($n^2 \geq 0$) in an atmospheric layer with constant N^2 and constant u_0 is

$$k^2 + l^2 \leq \beta/(u_0 - c) - f_0^2/4H^2N^2$$

This inequality must therefore apply to the stratosphere in the present case where $u_0 = u_0(h_1)$. Now it may be shown (cf. *Kuo* [1952]) that $c - u_0(0) = 0$ at the critical wavelength, and that its real part is positive for amplifying disturbances. Hence $u_0(h_1) - c = \Lambda h_1$ at the critical wavelength, and the condition for the above inequalities to be satisfied simultaneously is

$$\beta/2\Lambda H + \beta^2 N^2/4f_0^2\Lambda^2 < \beta/\Lambda h_1 - f_0^2/4N^2H^2$$

or

$$(f_0/2HN - \beta/2f_0\Lambda)^2 + (\beta/\Lambda H - \beta/\Lambda h_1) < 0$$

In the atmosphere $h_1 > H$, and $\beta/2f_0\Lambda \gg f_0/2NH$, $(\beta/\Lambda H)^{1/2}$. Therefore the inequality cannot be satisfied. It may be concluded that the unstable waves are external and cannot penetrate very far into the upper atmosphere.

If the zonal wind were to decrease with height above the tropopause but remain positive, upward propagation could take place. There is also the possibility that the upper atmosphere may itself be unstable, as in the case of the polar-night jet. Although these possibilities cannot be discounted, we shall confine the remainder of our analysis to forced stationary perturbations whose wavelengths may be sufficiently long to permit upward propagation. Such waves are limited laterally by the finite lateral extent of the perturbing forces as well as of the zonal current which, in reality, is not uniform but has a maximum in middle latitudes and decreases to the north and south. The mathematical analysis of currents with horizontal as well as vertical shear presents great difficulties. For the present we shall provide for the lateral variation only by selecting the predominant Fourier component of the perturbing forces, as in equation 4.2. As it happens, the y half-wave-length of this component corresponds fairly well to the lateral dimension of the zonal current. In the following analysis one might think of the currents as being confined by vertical rigid walls at $y = \pm\pi/l$, since no stationary oscillation is possible where u_0 vanishes.

5. Boundary Conditions

Ground condition. We shall assume here that

$$w' = W_0 e^{ikx} \cos ly \qquad (z = 0)$$

for some vertical velocity W_0. Its causes, such as variable elevation of the ground and heating near the surface, are immaterial.

When $W_0 = 0$ there are free oscillations. Then the ground and upper boundary conditions are each linear and homogeneous. Therefore there is a problem of hydrodynamical stability to find the eigenvalues c, and hence the growth rates of the waves. From equation 2.18 we obtain the boundary condition

$$(u_0 - c)\frac{dV}{dz} - \frac{du_0}{dz}V = 0 \qquad (z = 0)$$

When $W_0 \neq 0$, a forced oscillation with $c = 0$ is imposed by the ground condition. There is a steady solution, representing standing waves, for each pair of wave numbers unless resonance with a free oscillation occurs. On such a steady wave may be superposed the small-amplitude instabilities of other wave numbers, i.e. the free oscillations which are not stable. As these instabilities grow, nonlinear effects become significant, and the instabilities may dominate the steady waves or merge with the mean flow. When the steady waves are not obscured by instabilities we may use the condition

$$u_0\frac{dV}{dz} - \frac{du_0}{dz}V = -f_0\frac{W_0}{N^2} \qquad (z = 0)$$

Interfacial conditions. Discussion of the solution of the wave equation suggested the use of layers at whose interfaces u_0 and H or their derivatives are discontinuous. The solutions for V in the layers can be joined together by the boundary conditions that the normal velocity and pressure are continuous (these physical conditions could be alternatively proved by the theory of the differential equation when n^2 tends to a discontinuous function). Let an interface have equation

$$z = h \equiv \bar{h}(y) + h'$$

where

$$h'(x, y, t) = h_0 e^{ik(x-ct)} \cos ly$$

First the mean flow must satisfy the boundary conditions at the mean interface. The mean normal velocity is zero, and is therefore continuous at the interface. In order that the mean pressure $\bar{p} = R\bar{\rho}\bar{T}$ be continuous,

$$[\bar{\rho}H] = 0 \qquad (z = \bar{h})$$

where square brackets are used to denote the difference of their contents across the interface. The slope of the interface is implied by the relation $[\delta\bar{p}] = 0$, where the increment δ comes from any small translation in the interface. This leads to Margules' formula

$$d\bar{h}/dy = -f_0[\bar{\rho}u_0]/g[\bar{\rho}] \qquad (5.1)$$

unless $[\bar{\rho}] = 0$. If $[\bar{\rho}] = 0$, and the interface is not vertical, then $[u_0] = 0$. In this case $[\delta\bar{p}] = 0$ and the second difference $[\delta^2\bar{p}] = 0$ may be used to get

$$d\bar{h}/dy = -f_0u_0/g$$

The condition that the perturbed interface is a material surface is ? this may be a !

$$D(z - h)/Dt = 0 \qquad (5.2)$$

comment

If $u_0 \neq c$, this gives

$$h' = (w' - v'\, d\bar{h}/dy)/ik(u_0 - c) \qquad (z = \bar{h})$$

to first order. For continuity of the normal velocity, h' must be the same on each side of the interface, i.e.

$$[(w' - v'\, d\bar{h}/dy)/(u_0 - c)] = 0 \qquad (5.3)$$

For continuity of pressure,

$$\begin{aligned} 0 &= [\bar{p} + p'] && (z = h) \\ &= [h'\, \partial\bar{p}/\partial z + p'] && (z = \bar{h}) \\ &= -gh'[\bar{\rho}] + f_0[\bar{\rho}v']/ik \end{aligned}$$

With the use of the kinematic condition and a little algebra, it can be shown that this yields

$$[\bar{\rho}w'] = 0 \qquad (z = \bar{h}) \qquad (5.4)$$

The slope of the mean interface is neglected after the conditions have been derived. This may be justified by the conditions in section 2.

If $[\bar{\rho}]/\bar{\rho} \ll 1$ we may approximate the dynamic condition by

$$[w'] = 0 \qquad (z = \bar{h}) \qquad (5.5)$$

If $u_0 = c$ in a layer, the wave equation gives $V = 0$. Therefore $W = 0$. At the bottom of such a layer the boundary condition gives

$$w' = 0 \qquad h' = -v'(d\bar{h}/dy)/ik(u_0 - c)$$

Thus the layer acts as a rigid horizontal ceiling, although the interface really slopes. This totally reflects energy, as might have been expected on account of the fact that $n \to \pm i\,\infty$ as $u_0 - c \to 0$ through negative values. The perturbation pressure varies at the sinusoidal interface, but only nonlinear disturbances are generated above it.

this area is unclear.

Upper boundary condition. We deduce the upper boundary condition from the physical assumption that no wave component may propagate energy downward at infinity. This is the Sommerfeld radiation condition. (See *Eliassen and Palm* [1954] for its application to the similar problem of gravity waves.) If the vertical energy propagation is necessarily zero (as for external waves), the kinetic energy density at infinity must be bounded in a physically admissible solution.

The ground condition and the equation are symmetric in time, there being periodic or steady motion of an inviscid fluid. The time asymmetry of the upper condition is necessary to permit any net energy propagation at all. It is customary to regard waves as the asymptotic time limit of a component of an initial disturbance, or as the inviscid limit of a wave in a slightly viscous fluid. The former alternative has been used for diverse wave motions and has confirmed the radiation condition in each case. The radiation condition has also been confirmed by use of Rayleigh's 'fictitious viscosity' (a simplified viscosity that permits energy dissipation without raising the order of the equations of motion) in the inviscid limit.

We shall verify the radiation condition for baroclinic waves in an atmosphere of constant velocity and temperature by use of a real horizontal viscous stress, which acts as a Rayleigh fictitious stress. We shall then take such a uniform atmosphere above a certain height and use the interfacial conditions to join together the solution in the region below, where u_0 and H may vary. This will give in all two boundary conditions (one at the ground, one at the top interface) linear in V and dV/dz for the second-order wave equation in V.

This artificial ~~use of a uniform upper layer,~~ where the real atmosphere is by no means uniform, is justified if all energy reaching that layer is absorbed in it and not reflected down. The model solution will then describe faithfully the real flow below but not above the top interface. We add the proviso that the top interface be below the region where mechanisms, such as viscosity and hydromagnetic heating, not in our model are important. For a perfect gas the mean kinematic viscosity is proportional to $\bar{T}^{1/2}\bar{p}^{-1}$, so that the viscous dissipation per unit volume in planetary waves is proportional to

$$\bar{T}^{1/2}[(\partial u/\partial z)^2 + (\partial v/\partial z)^2], \quad \text{i.e. to } \bar{p}^{-1}\bar{T}^{1/2},$$

which increases rapidly with height. We estimate that viscous dissipation becomes important above 100 km or so, according to the wavelengths. The dissipative effect on the large-scale motions of small-scale eddies is more difficult to gage. Ohmic dissipation by hydromagnetic effects occurs higher up.

To deduce the radiation condition, take the equation for baroclinic waves in a uniform atmosphere with horizontal viscous stress coefficient K (cf. *Kuo* [1952])

$$\frac{d^2 V}{dz^2} - \frac{1}{H}\frac{dV}{dz} + \left[\frac{(k^2 + l^2)(u_c - u_0 + c)N^2}{(u_0 - c)f_0^2} + \frac{i(k^2 + l^2)KN^2}{k(u_0 - c)f_0^2}\right]V = 0$$

Therefore

$$V = (Ae^{iqz} + Be^{-iqz})e^{z/2H}$$

where

$$q^2 \equiv n^2 + i(k^2 + l^2)^2 KN^2/kf_0^2(u_0 - c)$$

and n is as previously defined for an inviscid fluid. Therefore

$$q = h\{1 + i(k^2 + l^2)^2 KN^2/2kn \cdot (u_0 - c)f_0^2 + 0(K^2)\}$$

for small K. It follows that $B = 0$ in order that the kinetic energy density Q_k be bounded at infinity. When $K \to 0$ this concurs with the Sommerfeld radiation condition.

To find the upper boundary condition, we join the solution $V_J = A_J\ e^{(1+i\nu_J)z/2H}$ above the interface ($z = h_J$, say) with the solution below. (We use the subscript J for quantities in the upper layer; and define $\nu_J - 2Hn_J$.) On elimination of A_J from the boundary conditions, we find

$$\left(u_0 \frac{dV}{dz} - \frac{du_0}{dz} V\right)\left\{\frac{2H_J N_J^2}{1 + in_J} + \frac{fu_{0J}}{d\bar{h}/dy} \cdot \left(\frac{\bar{\rho}}{\bar{\rho}_J} - \frac{u_{0J}}{u_0}\right)\right\} - \frac{N^2 u_{0J}^2}{u_0} V = 0 \quad (z = \bar{h}_J)$$

upper b.c. for both layer

6. Standing Waves in Layer Atmospheres

Many-layer atmosphere. Suppose that the atmosphere is isothermal with piecewise constant velocity; i.e., suppose that

$$u_0 = u_{0j} \qquad \nu = \nu_j \equiv 2Hn_j \qquad (h_{j-1} < z < h_j;\ j = 1, 2, \cdots, J)$$

with constant H, N^2 throughout the atmosphere in accordance with the neglect of inertial effects of temperature variation. Then the general steady solution of the wave equation in the jth layer may be written as

$$V_j = (D_j \sin n_j z + E_j \cos n_j z)e^{z/2H} \qquad (j = 1, 2, \cdots, J - 1)$$

whether n_j is real or pure imaginary. In the upper layer,

$$V_J = A_J e^{z/2H + in_J(z - h_J)}$$

in order to satisfy the condition of upward energy flux or bounded energy density.

The ground condition gives

$$d_{10}\ D_1 + e_{10}E_1 = -2HN^2 W_0/f_0 u_{01} \equiv Q_0 \ (R_1)$$

say, where

$$C_{jk} \equiv \cos n_j \bar{h}_k \qquad S_{jk} \equiv \sin n_j \bar{h}_k$$

$$d_{jk} \equiv \nu_j C_{jk} + S_{jk} \qquad e_{jk} \equiv -\nu_j S_{jk} + C_{jk}$$

The approximate dynamic interfacial condition, $[w'] = 0$, gives

$$r_j(d_{jj}\ D_j + e_{jj}E_j) - (d_{j+1,j}\ D_{j+1} + e_{j+1,j}E_{j+1}) = 0 \quad (j = 1, 2, \cdots J - 1) \qquad (R_{2j})$$

where

$$r_j \equiv u_{0j}/u_{0,j+1}$$

The kinematic interfacial condition, after a little simplification with the aid of the dynamic condition, gives

$$(d_{jj} - q_j r_j^{-1} S_{jj})\ D_j + (e_{jj} - q_j r_j^{-1} C_{jj})E_j + q_j(S_{j+1,j}\ D_{j+1} + C_{j+1,j}E_{j+1}) = 0 \quad (j = 1, 2, \cdots, J - 1) \qquad (R_{2j+1})$$

where

$$q_j \equiv 2HN^2/g\eta_j = -2HN^2(d\bar{h}_j/dy)/f_0(1 - r_j)u_{0,j+1}$$

and

$$\eta_j \equiv (\bar{\rho}_{j+1} - \bar{\rho}_j)/\bar{\rho}$$

The upper boundary condition is

$$(d_{J-1,J-1} - q_{J-1}r_{J-1}^{-1}S_{J-1,J-1})\ D_{J-1} + (e_{J-1,J-1} - q_{J-1}r_{J-1}^{-1}C_{J-1,J-1})E_{J-1} + q_{J-1}A_J = 0 \qquad (R_{2J-1})$$

The conditions R_k ($k = 1, 2, \cdots, 2J - 1$) are an inhomogeneous system of $(2J - 1)$ linear algebraic equations to find the $(2J - 1)$ unknowns $D_1, E_1, D_2, \cdots, A_J$. There is a unique solution unless the discriminant vanishes, in which case there is a free mode with eigenvalue $c = 0$.

One-layer atmosphere. For a one-layer atmosphere we at once get

$$V = \frac{Q_0}{1 + i\nu} e^{(1+i\nu)z/2H} \qquad (z \geq 0)$$

Therefore the upward energy flux

$$W_z = \frac{\nu HN^2 W_0^2(\bar{\rho}e^{z/H})}{ku_0(1 + \nu^2)}$$

for real ν (i.e. for internal waves) and is zero for imaginary ν (i.e. for external waves). Also the kinetic energy density

$$Q_k = \frac{(1 + k^2/l^2)H^2N^4W_0^2(\bar{\rho}e^{z/H})}{f_0^2 u_0^2} \times \begin{cases} 1 & (n \text{ real}) \\ e^{2inz} & (n \text{ imaginary}) \end{cases}$$

Two-layer atmosphere. In this case ($J = 2$) the equations are

$$d_{10}\ D_1 + e_{10}E_1 = Q_0$$

$$r(d_{11}\ D_1 + e_{11}E_1) - (1 + i\nu_2)A_2 = 0$$

and

$$(d_{11} - qr^{-1}S_{11})\ D_1 + (e_{11} - qr^{-1}C_{11})E_1 + qA_2 = 0$$

Their solution is

$$D_1/Q_0 = (e_{11}t - qr^{-1}C_{11})/\Delta'$$

$$E_1/Q_0 = -(d_{11}t - qr^{-1}S_{11})/\Delta'$$

$$A_2/Q_0 = -\nu_2 q/\Delta$$

where the discriminant

$$\Delta \equiv (1 + i\nu_2)\Delta'$$

$$\Delta' \equiv d_{10}(e_{11}t - qr^{-1}C_{11}) - e_{10}(d_{11}t - qr^{-1}S_{11})$$

$$t \equiv 1 + qr/(1 + i\nu_2)$$

We can calculate the transmission coefficient of energy flux as follows. The complex amplitude A_1 of the incident wave (with meridional velocity varying as $e^{(in_1+1/2H)z}$) is $\frac{1}{2}(E_1 - iD_1)$, that of the reflected wave, B_1 (varying as $e^{(-in_1+H/2)z}$) is $\frac{1}{2}(E_1 + iD_1)$, and that of the transmitted wave (varying as $e^{(in_2+H/2)z}$) is $A_2e^{-in_2h}$. It can be seen that

$$A_1 : B_1 : A_2e^{-in_2h} = -ie^{-in_1h}\{(1 - i\nu_1)\cdot(1 + i\nu_2 + qr) - qr^{-1}(1 + i\nu_2)\} : ie^{in_1h}\{(1 + i\nu_1)(1 + i\nu_2 + qr) - qr^{-1}(1 + i\nu_2)\} : -2\nu_2 qA_2e^{-in_2h}$$

If ν_2 is pure imaginary, the upper layer is a perfect reflector, $|A_1|^2 = |B_1|^2$, and $W_z = 0$ everywhere. In fact, the transmission coefficient, the ratio of the energy flux of the transmitted wave to that of the incident wave,

$$T = \begin{cases} 0 & (\nu_2 \text{ imaginary}) \\ \nu_2 \, |A_2 e^{-in_2h}|^2 \, N_1^2 / r\nu_1 \, |A_1|^2 \, N_2^2 & (\nu_1, \nu_2 \text{ real}) \\ \nu_2 |A_2 e^{-in_2h}|^2 N_1^2 / r\nu_1 (A_1 B_1^* - A_1^* B_1) N_2^2 & (\nu_2 \text{ real}, \nu_1 \text{ imaginary}) \end{cases}$$

$$= \left\{ \begin{array}{c} 0 \\ 4N_1^2\nu_2^3 q^2 / rN_2^2\nu_1 \{[1 + qr - qr^{-1} \\ + \nu_1\nu_2]^2 + [\nu_1 - \nu_2 + qr\nu_1 + qr^{-1}\nu_2]^2 \\ - N_1^2\nu_2^2 / rN_2^2\nu_1^2 \end{array} \right\}$$

respectively.

Two-layer atmosphere with rigid top. If there is a three-layer atmosphere with $u_0 = 0$ in the upper layer, the boundary conditions can be shown to give

$$d_{10} \, D_1 + e_{10}E_1 = Q_0$$

$$r(d_{11} \, D_1 + e_{11}E_1) - (d_{21} \, D_2 + e_{21}E_2) = 0$$

$$(d_{11} - qr^{-1}S_{11}) \, D_1 + (e_{11} - qr^{-1}C_{11})E_1 + q(S_{21} \, D_2 + C_{21}E_2) = 0$$

$$d_{22} \, D_2 + e_{22}E_2 = 0$$

The solution is

$$D_1/Q_0 = t/\Delta \qquad E_1/Q_0 = u/\Delta$$

$$D_2/Q_0 = -e_{22}\nu_1 q/\Delta \qquad E_2/Q_0 = d_{22}\nu_1 q/\Delta$$

where the discriminant

$$\Delta \equiv q_0 t + b_{10} u$$

$$t \equiv e_{22}[(qrS_{21} + d_{21})e_{11} - qr^{-1} \, d_{21}C_{11}] - d_{22}[(qrC_{21} + e_{21}) \, d_{11} - qr^{-1}e_{21}C_{11}]$$

$$u \equiv -e_{22}[(qrS_{21} + d_{21}) \, d_{11} - qr^{-1} \, d_{21}S_{11}] + d_{22}[(qrC_{21} + e_{21}) \, d_{11} - qr^{-1}e_{21}S_{11}]$$

Three-layer atmosphere. When $J = 3$, we find

$$d_{10} \, D_1 + e_{10}E_1 = Q_0$$

$$r_1(d_{11} \, D_1 + e_{11}E_1) - (d_{21} \, D_2 + e_{21}E_2) = 0$$

$$(d_{11} - q_1r_1^{-1}S_{11}) \, D_1 + (e_{11} - q_1r_1^{-1}C_{11})E_1 + q_1(S_{21} \, D_2 + C_{21}E_2) = 0$$

$$r_2(d_{22} \, D_2 + e_{22}E_2) - (1 + i\nu_3)A_3 = 0$$

$$(d_{22} - q_2r_2^{-1}S_{22}) \, D_2 + (e_{22} - q_2r_2^{-1}C_{22})E_2 + q_2A_3 = 0$$

The solution is

$$D_1 = \{e_{11}Q_0 - e_{10}r_1^{-1}(d_{21} \, D_2 + e_{21}E_2)\}/\Delta'$$

$$E_1 = -\{d_{11}Q_0 - d_{10}r_1^{-1}(d_{21} \, D_2 + e_{21}E_2)\}/\Delta'$$

$$D_2 = -\nu_1 q_1(te_{22} - q_2r_2^{-1}C_{22})Q_0/\Delta$$

$$E_2 = \nu_1 q_1(t \, d_{22} - q_2r_2^{-1}S_{22})Q_0/\Delta$$

$$A_3 = \nu_1\nu_2 q_1 q_2 Q_0/\Delta(1 + i\nu_3)$$

where the discriminant

$$\Delta \equiv \Delta' \, \Delta''$$

$$\Delta' \equiv -(1 + \nu_1^2)S_{11}$$

$$\Delta'' \equiv (d_{21}u + q_1r_1S_{21})(e_{22}t - q_2r_2^{-1}C_{22}) - (e_{21}u + q_1r_1C_{21})(e_{22}t - qr_2^{-1}S_{22})$$

$$t \equiv 1 + q_2r_2/(1 + i\nu_3)$$

$$u \equiv 1 + q_1(e_{10}S_{11} - d_{10}C_{11})/r_1 \, \Delta'$$

Two-layer atmospheres with shear. We shall also use the exact solution of the wave equation for an atmosphere with constant N^2 and linear zonal velocity. We shall take two two-layer models of the stratosphere and troposphere, the velocity vanishing at the ground and being continuous in each. In the first, the velocity has constant shear in the troposphere and is constant above the troposphere. In the other the shear is constant, positive below and negative above the troposphere.

In each case the boundary conditions are: (*a*) $V = N_T^2W_0/f_0\Lambda_T \equiv R_0$, say (ground, $z = 0$); (*b*) $[V]$, $[W] = 0$ (tropopause, $z = h$); (*c*) the Sommerfeld radiation condition at infinity if energy propagation is possible there, otherwise the boundedness of the kinetic energy density.

(i)

$$H = \begin{cases} H_T \\ H_S \end{cases} \qquad N = \begin{cases} N_T \\ N_S \end{cases}$$

$$u_0 = \begin{cases} \Lambda z & (z < h) \\ \Lambda h & (z > h) \end{cases}$$

In this case

$$V_T = e^{(1/2H_T - \bar{a})z}\psi_T, \ \psi_T(\xi; r)$$
$$= R_0\{\psi_1 + D\psi_2\} \qquad (z < h)$$
$$V_S = R_0 A_S e^{(1+i\nu_S)z/2H_S} \qquad (z > h)$$

After some algebra it can be shown that

$$D = \begin{cases} -(P_1 + Q_1)/(P_2 + Q_2) \\ \qquad\qquad (\text{imaginary } \nu_S) \\ [-(P_1P_2 + Q_1Q_2) \\ \quad + (Q_1P_2 - Q_2P_1)i]/(P_2^2 + Q_2^2) \\ \qquad\qquad (\text{real } \nu_S) \end{cases}$$

where

$$P_i \equiv 2\bar{a}\psi_i' - \{\bar{a} - \tfrac{1}{2}H_T^{-1} + \tfrac{1}{2}(N_T/N_S)^2 H_S^{-1} + h^{-1}\}\psi_i$$
$$Q_i \equiv \tfrac{1}{2}|n_S|(N_T/N_S)^2\psi_i \qquad (z = h, i = 1, 2)$$

Also

$$A_S/R_0 = e^{-(\bar{a}+in_S)h} \cdot \{\psi_1 + D\psi_2\}_{z=h} e^{h/2(H_T^{-1} - H_S^{-1})}$$

(ii)

$$H = \text{constant} \qquad N = \begin{cases} N_T \\ N_S \end{cases}$$

$$u_0 = \begin{cases} \Lambda_T z & (z \le h, \Lambda_T > 0) \\ \Lambda_T h + \Lambda_S(z - h) & (z \ge h, \Lambda_S < 0) \end{cases}$$

Here

$$\psi_T = R_0\{\psi_1(\xi_T, r_T) + D_T\psi_2(\xi_T, r_T)\}$$

and

$$\psi_S = D_S R_0 \psi_1(\xi_S, r_S)$$

It can be shown that

$$D_T = -\left\{\left(\frac{2\bar{a}}{N^2}\psi_1' - b\psi_1\right)\Big/\left(\frac{2\bar{a}}{N^2}\psi_2' - b\psi_2\right)\right\}_{T, z=h}$$

and

$$D_S = e^{(\bar{a}_S - \bar{a}_T)h}\{\psi_{1T} + D_T\psi_{2T})/\psi_S\}_{z=h}$$

where

$$b \equiv \frac{2\bar{a}_S}{N_S^2}\frac{\psi_S'}{\psi_S} + \left[\frac{\Lambda/u_0 + 1/2H - \bar{a}}{N^2}\right]_T^S$$

When $z \to \infty$, $\xi_S > 0$, and we use the principal value of the logarithm in ψ_{1s}. As z decreases to $< h$, ξ_S becomes negative, and one of the branches of the logarithm must be chosen. To find the correct branch we take the limit of an amplified disturbance (i.e., one with $c_i > 0$) as $c_i \to 0$, the rule established for hydrodynamical stability [cf. *Lin*, 1955; *Kuo*, 1952] to get the proper inviscid limit. Consider a slightly amplified solution of the time-dependent equation with $\xi = 2\bar{a}(u_0 - c)/\Lambda$ and $c = c_r + ic_i$ $(c_i > 0)$. As z increases through the zero of $u_0 - c_r$ the argument of ξ in the complex plane decreases from nearly π to nearly 0 in the upper half of the complex plane, since $\xi = \xi_r + i\xi_i$, where $\xi_i = -2\bar{a}c_i/\Lambda$, and Λ is negative. Hence the proper branch of the logarithm is

$$\ln \xi_S = \ln |\xi_S| + \pi i \qquad (\xi_S < 0)$$

This selection is confirmed by the condition that the energy flow in the stratosphere be upward for small Λ. Thus we find that

$$\text{sign}\, \overline{p'w'} = \mp \text{sign}\, r_S\Lambda_S\xi_S$$

according as $\ln \xi_S = \ln |\xi_S| \pm \pi i$. If $|\Lambda_S|$ is not so large that r is positive we obtain confirmation. If $|\Lambda_S|$ is so large that r is positive, there will be a conversion of potential energy in the stratosphere and downward energy propagation below the zero of ξ. Thus we take

$$\psi_S(r_S, \xi_S) = \begin{cases} \psi_1(r_S; \xi_S) & (\xi_S > 0) \\ \psi_1(r_S; \xi_S) - \pi i\psi_2(r_S; \xi_S) & (\xi_S < 0) \end{cases}$$

where ψ_1 is computed with the principal value of the logarithm it contains.

7. Some Numerical Solutions of Layer Atmospheres

In section 6 we gave a number of models of layer atmospheres. Subject to mathematical tractability and simplicity, we chose the models to represent aspects of the real atmosphere most relevant to our study. Here we summarize a few numerical results with parameters typical of temperature and velocity structure at various heights and seasons. We hope to gain a view of the whole by study of these details.

In all models, $\gamma \equiv c_p/c_v = 1.4$, $g = 9.8$ m sec^{-2}, $\beta = 1.6 \times 10^{-11}$ m^{-1} sec^{-1}, and $f_0 = 10^{-4}$ sec^{-1}.

Two-layer atmospheres with no shear. In each case

$$H = 7.07 \times 10^3 \text{ m},$$

$$N^2/f_0^2 = 3.96 \times 10^{-4},$$

$$\eta = -0.1,$$

$$q \equiv 2HN^2/g\eta = -5.71,$$

$$h = 10^4.$$

	(i)	(ii)	(iii)
u_{01} in m sec^{-1} =	22.5	7.5	10
u_{02} in m sec^{-1} =	7.5	22.5	50
$L \equiv 2\pi(k^2+l^2)^{-1/2}$ in m =	6×10^6	6×10^6	10^7

From these parameters we compute the following:

	(i)	(ii)	(iii)
ν_1 =	$2.01i$	2.69	2.92
ν_2 =	2.69	$2.01i$	$1.26i$
D_1/Q_0 =	$0.0195-1.03i$	0.258	-0.0769
E_1/Q_0 =	$-1.08\ -0.0391i$	0.307	1.22
A_2/Q_0 =	$-0.115\ -0.0361i$	0.284	2.10
T =	0.597	0	0

Two-layer atmosphere with rigid top. Take $u_{01} = 22.5$ m sec^{-1}, $u_{02} = 7.5$ m sec^{-1}, $h_1 = 10^4$ m, $h_2 = 3 \times 10^4$ m, $H = 7.07 \times 10^3$ m, $N^2/f_0^2 = 3.96 \times 10^4$, $\eta = -0.1$, so that

$$q = -5.71 \qquad L = 6 \times 10^6 \text{ m}$$

Then we compute the following: $\nu_1 = 2.01i$, $\nu_2 = 2.69$, $D_1/Q_0 = -1.14i$, $E_1/Q_0 = -1.28$, $D_2/Q_0 = 0.534$, $E_2/Q_0 = -0.390$.

Three-layer atmospheres. In each case $H = 7.07 \times 10^3$ m, $N^2f_0^2 = 3.96 \times 10^4$, $\eta_1 = -0.1$, $\eta_2 = -0.4$, $q_1 = -5.71$, $q_2 = -1.43$.

	(i)	(ii)
u_{01} in m sec^{-1} =	7.5	10
u_{01} in m sec^{-1} =	22.5	50
u_{03} in m sec^{-1} =	7.5	10
h_1 in m =	5×10^3	10^4
h_2 in m =	10^4	10^5
L in m =	6×10^6	10^7

From these parameters we compute the following:

	(i)	(ii)
ν_1 =	2.69	2.92
ν_2 =	$2.01i$	$1.26i$
D_1/Q_0 =	$-0.994+0.0421i$	-0.0778
E_1/Q_0 =	$3.67\ -0.113i$	1.23
D_2/Q_0 =	$2.38\ -0.897i$	$2.04-0.519i$
E_2/Q_0 =	$-0.890-2.48i$	$-0.519-2.04i$
A_3/Q_0 =	$0.113-0.246i$	$(1.51\ -5.96i)\times 10^{-5}$

Two-layer models with shear. (i) Take $u_0 = 3 \times 10^{-3}z$ m sec^{-1} for $z \leq 10^4$ m, $u_0 = 30$ m sec^{-1} for $z \geq 10^4$ m, $H_T = 6.69 \times 10^3$ m (giving $T_T = 228°K$) and $H_S = 6.21 \times 10^3$ m (giving $T_S = 212°K$). Then

$$N_T^2/f_0^2 = 1.69 \times 10^4$$

corresponding to a lapse rate of 6.8° per km in the troposphere, and

$$N_S^2/f_0^2 = 4.51 \times 10^4$$

corresponding to an isothermal stratosphere. Table 1 gives the computed values of $\bar{a}$, r, ν_S, and A_S for various values of L.

There is resonance with a free mode with eigenvalue $c = 0$ for $L \cong 7 \times 10^6$ m. Then D and A_S are infinite.

$\nu_S = 0$ when $L = 1.01 \times 10^7$ m. Below this wavelength, all waves are external, and so no vertical energy propagation is possible.

(ii) Take $u_0 = 3 \times 10^{-3}z$ m sec^{-1} for $z \leq 10^4$, $u_0 = 30 - 2.36 \times 10^{-3}\,(z - 10^4)$ for $z \geq 10^4$ m, $H = 7.07 \times 10^3$ m, $N_T^2 = \frac{1}{4} N_S^2 = 1.41 \times 10^{-4}$ sec^{-2}, and $L = 6.5 \times 10^6$ m. Then we find $\bar{a}_T = 1.35 \times 10^{-4}$ m^{-1}, $\bar{a}_S = 2.40 \times 10^{-4}$ m^{-1}, $r_T = 0.803$, $r_S = -0.502$, $D_T = 2.63 - 0.0133i$, $A_S = -3.98i + 0.0945$.

Discussion of numerical results. These numerical results confirm the general predictions of the analogies of the electromagnetic wave equation and Schrödinger's equation. When ν is real in a layer with constant u_0 and H, we see that V oscillates with z, having an amplitude of order $(|D|^2 + |E|^2)^{1/2}e^{z/H}$. When ν is pure imaginary, we find $D \cong iE$, so that V behaves like $Ee^{(1+i\nu)z/2H}$, approximately. The thicker the layer in which n is pure imaginary, the better is this approximation.

For our values of H, N^2, f_0,

$$Q_0 \equiv -2HN^2W_0/f_0u_{01}$$

$$\cong 5.6 \times 10^4 W_0/u_{01} \text{ m sec}^{-1}$$

to get the order of magnitude of the velocity in each model. It can be seen that this gives

$V \sim 5$ m sec^{-1} if $W_0 \sim 2 \times 10^{-3}$ m sec^{-1} at the ground.

In the two-layer atmospheres with no shear, waves of length 6000 km are transmitted where $u_0 = 7.5$ m sec^{-1} but reflected where $u_0 = 22.5$ m sec^{-1}. In case i the lower layer reflects about nine-tenths of the energy. In cases ii and iii there is no net vertical energy flux at any height because the infinitely thick top layers are reflectors. The kinetic energy density in the upper layer decreases by a factor of e in half a scale height in case ii and a whole one in case iii. Although the velocity in the upper layer in case iii is more than twice as great as in case ii (50 m sec^{-1} against 22.5 m sec^{-1}), the greater wavelength chosen (10^7 m against 6×10^6 m) permits more energy in case iii.

In the two-layer atmosphere with a rigid ceiling, there is again no vertical energy flux. The amplitude of the kinetic energy density in the lower layer is about double that in the upper.

Both the selected three-layer atmospheres permit vertical energy flux. However, the thick reflecting layer where $u_0 = 50$ m sec^{-1} in case ii permits less than 10^{-8} of the energy released at the ground to penetrate to infinity.

In the thick layer (10^4 to 10^5 m) of case ii, we find $D_2 = iE_2$ to the accuracy of the calculation. This shows that the descending exponential is picked out. This approximation is not so good for the thinner layer (5×10^3 to 10^4 m) of case i, although the negative square of the 'refractive index' is greater in magnitude there.

Of the models with shear, case i permits vertical energy flux for $L > 10^7$ m, but ii permits none. The results for case i are quite extensive because it is a fair model of the troposphere and stratosphere.

If the mean velocity and temperature are uniform in the upper layer, we can compute the energy flux

$$W_z = \tfrac{1}{2} f_0^2 N^{-2} \bar{\rho}(0) k^{-1} n \, |A|^2 \, u_0$$

where n is real, $\bar{\rho}(0)$ the constant $\bar{\rho}\, e^{z/H}$, and A equals $A_S Q_0$ or $A_S R_0$, as is appropriate for the case. In case i of the two-layer atmospheres with shear, this gives $kW_z/W_0^2 \sim 500$ g m^{-3} sec^{-1}. If $L \sim 10^7$ m and $W_0 \sim 2 \times 10^{-3}$ m sec^{-1}, this gives $W_z \sim 4 \times 10^3$ g sec^{-3} = 4 joule m^{-2} sec^{-1}, the order of the tropospheric dissipation of energy. Thus it appears that in some circumstances the energy loss by mechanical wave propagation into the upper atmosphere may be comparable to the energy loss by frictional dissipation at the ground.

8. Nonlinear Theory

Momentum and heat equations. When energy is propagated into the upper atmosphere, the velocity becomes large and nonlinear effects become important. We shall consider these effects in this section. To derive the equation for zonally averaged flow, we average equations 2.12 and 2.10. After some manipulation and use of the averaged continuity equation

$$\partial(\bar{\rho} v_0)/\partial y + \partial(\bar{\rho} w_0)/\partial z = 0$$

TABLE 1

L, km	$\bar{a}$, km^{-1}	r	ν_S	D	A_S
∞	0.9750	1.6	1.64	$2.85 - 1.61i$	$-2.28 + 0.30i$
2.93×10^4	0.800	1.5	1.54	$1.87 - 1.60i$	$-2.68 + 0.01i$
1.97×10^4	0.0857	1.4	1.41	$1.08 - 1.59i$	$-2.78 - 0.34i$
1.54×10^4	0.0923	1.3	1.23	$0.24 - 1.57i$	$-2.96 - 0.86i$
1.23×10^4	0.100	1.2	0.950	$-1.67 - 1.35i$	$-2.68 - 1.69i$
1.03×10^4	0.109	1.1	0.358	$-6.60 - 0.67i$	$-1.40 - 3.26i$
8.72×10^3	0.120	1.0	$0.358i$	14.9	-3.94
7.41×10^3	0.133	0.9	$1.51i$	51.7	-9.65
6.29×10^3	0.150	0.8	$2.06i$	-1.75	7.44×10^{-1}
5.30×10^3	0.171	0.7	$2.66i$	-0.85	2.50×10^{-1}
4.41×10^3	0.200	0.6	$3.38i$	-0.36	7.40×10^{-2}
3.58×10^3	0.240	0.5	$4.32i$	-0.10	1.61×10^{-2}
2.81×10^3	0.300	0.4	$5.60i$	-0.021	2.20×10^{-3}
2.08×10^3	0.400	0.3	$7.80i$	-0.0022	8.39×10^{-5}

we get

$$-\partial^2\chi_0/\partial y\,\partial t = -\partial M/\partial y + f_0 v_0 \tag{8.1}$$

$$\partial^2\chi_0/\partial z\,\partial t = -\partial B/\partial y - f_0^{-1}N^2 w_0 \tag{8.2}$$

where

$$M \equiv -\overline{\frac{\partial\chi'}{\partial x}\frac{\partial\chi'}{\partial y}} = \overline{u'v'} \tag{8.3}$$

is the eddy momentum flux per unit mass in the y direction,

$$B \equiv \overline{\frac{\partial\chi'}{\partial x}\frac{\partial\chi'}{\partial z}} \tag{8.4}$$

is proportional to the eddy heat flux in the y direction, and the bars denote x averages. Eliminating v_0 and w_0 by means of the averaged continuity equation, we get

$$\left[\frac{\partial^2}{\partial y^2} + f_0^2\left(\frac{\partial}{\partial z} - \frac{1}{H}\right)\left(\frac{1}{N^2}\frac{\partial}{\partial z}\right)\right]\frac{\partial\chi_0}{\partial t}$$

$$= -\frac{\partial^2 M}{\partial y^2} - f_0^2\left(\frac{\partial}{\partial z} - \frac{1}{H}\right)\left(\frac{1}{N^2}\frac{\partial B}{\partial y}\right) \tag{8.5}$$

Boundary conditions. The boundary condition at the ground is obtained by averaging $w = \mathbf{v}_h\cdot\nabla h$. Thus

$$w_0 = -\overline{\frac{\partial\chi'}{\partial y}\frac{\partial h}{\partial x}} + \overline{\frac{\partial\chi'}{\partial x}\frac{\partial h}{\partial y}} \tag{8.6}$$

But, from equations 2.15 and 4.1,

$$u_0\frac{\partial h}{\partial x} = -\frac{f_0}{N^2}\left(u_0\frac{\partial^2\chi'}{\partial x\,\partial z} - \frac{\partial u_0}{\partial z}\frac{\partial\chi'}{\partial x}\right)$$

and therefore

$$h = -\frac{f_0}{N^2}\left(\frac{\partial\chi'}{\partial z} - \frac{\chi'}{u_0}\frac{\partial u_0}{\partial z}\right)$$

Hence equation 8.6 becomes

$$w_0 = \frac{f_0}{N^2}\left(\overline{\frac{\partial\chi'}{\partial y}\frac{\partial^2\chi'}{\partial x\,\partial z}} - \overline{\frac{\partial\chi'}{\partial x}\frac{\partial^2\chi'}{\partial y\,\partial z}}\right) = -\frac{f_0}{N^2}\frac{\partial B}{\partial y}$$

and by substitution in (8.2) we obtain

$$\frac{\partial}{\partial z}\left(\frac{\partial\chi_0}{\partial t}\right) = 0 \qquad (z = 0) \tag{8.7}$$

The boundary condition at a surface of discontinuity is obtained by the methods of section 5 as follows. The interface has normal

$$\mathbf{n} \equiv (\partial h/\partial x,\ \partial h/\partial y,\ -1).$$

In steady flow the velocity is tangential to the material interface; therefore $\mathbf{n}\cdot\mathbf{v} = 0$, and $\mathbf{n}\cdot[\mathbf{v}] = 0$ a fortiori. Also the pressure is continuous at the interface. Therefore the component of the pressure gradient tangential to the interface is continuous. But $\mathbf{v}$ is directed along a tangent. Therefore

$$0 = [\mathbf{v}]\cdot[\text{grad } p] = [\mathbf{v}]\cdot[(f_0\bar{\rho}v,\ -f_0\bar{\rho}u,\ -g\bar{\rho})]$$

These two conditions give

$$[w] = [u]\,\partial h/\partial x + [v]\,\partial h/\partial y$$

$$[w] = f_0\{[u][\bar{\rho}v] - [\bar{\rho}u][v]\}/g[\bar{\rho}]$$

$$= f_0(u_2v_1 - u_1v_2)/g$$

identically. The averages of these conditions are

$$[w_0] = -\frac{\partial}{\partial y}\left(\overline{[\chi']\frac{\partial h'}{\partial x}}\right) + \frac{\partial\bar{h}}{\partial y}[\bar{v}]$$

$$= -\frac{\partial}{\partial y}\left(\overline{[\chi']\frac{\partial h'}{\partial x}}\right) - \frac{f_0}{g}\frac{[\bar{\rho}u_0]}{[\bar{\rho}]}[v_0] \tag{8.8}$$

$$[w_0] = f_0(\overline{u_2'v_1'} - \overline{u_1'v_2'})/g \tag{8.9}$$

(We may not assume in the kinematic condition that the zonal mean of v is geostrophic, since there will be a net nongeostrophic meridional circulation which will tilt the surface of discontinuity.) The perturbation forms of the boundary conditions are

$$[w'] = [\bar{u}]\frac{\partial h'}{\partial x} - \frac{f_0}{g}\frac{[\bar{\rho}\bar{u}]}{[\bar{\rho}]}\left[\frac{\partial\chi'}{\partial x}\right]$$

$$[w'] = f_0(\bar{u}_2v_1' - \bar{u}_1v_2')/g$$

from which we obtain

$$\partial h'/\partial x = f_0[\bar{\rho}v']/g[\bar{\rho}]$$

With this equation one may show that

$$\frac{f_0}{g}(\overline{u_2'v_1'} - \overline{u_1'v_2'})$$

$$= -\frac{\partial}{\partial y}\left(\overline{[\chi']\frac{\partial h'}{\partial x}}\right) = -\frac{f_0}{g}\frac{\partial\Pi}{\partial y}$$

where

$$\Pi \equiv \overline{\chi_2'\,\partial\chi_1'/\partial x}$$

Then equations 8.7, 8.8, 8.9 give

$$[v_0] = 0 \qquad [w_0] = -f_0 g^{-1}\,\partial\Pi/\partial y \tag{8.10}$$

Substitution into equations 8.1 and 8.2 gives finally

$$\left.\begin{aligned}\frac{\partial}{\partial t}\left[\frac{\partial \chi_0}{\partial y}\right] &= -\left[\frac{\partial M}{\partial y}\right]\\ \frac{\partial}{\partial t}\left[\frac{g}{N^2}\frac{\partial \chi_0}{\partial z}\right] &= -\left[\frac{g}{N^2}\frac{\partial B}{\partial y}\right] + \frac{\partial \Pi}{\partial y}\end{aligned}\right\}$$

$$(z = h) \qquad (8.11)$$

At infinity we require the mean flow to be bounded.

If the surface of discontinuity is one at which the vertical gradients of u_0 and $\bar{p}$ are discontinuous but u_0 and ρ are themselves continuous, $[\chi'] = [\partial\chi'/\partial x] = [w'] = 0$, and $\Pi = 0$. Multiplication of equation 2.15 by χ'/N^2 and integration in x then give $[B/N^2] = 0$. Furthermore, $M = 0$, since we have chosen waves with $\chi' \propto e^{ikx} \cos ly$, so that $\partial\chi'/\partial x$ and $\partial\chi'/\partial y$ are 90° out of phase. Thus the surface and interfacial conditions become homogeneous.

Consider now the right-hand side of equation 8.5. Because of the vanishing of M this may be written

$$-f_0^2\left(\frac{\partial}{\partial z} - \frac{1}{H}\right)\left(\frac{1}{N^2}\frac{\partial B}{\partial y}\right) = -\frac{f_0^2}{\bar{p}}\frac{\partial^2}{\partial y\,\partial z}\left(\frac{\bar{p}B}{N^2}\right)$$

Multiplication of equation 2.15 by $f_0p'/N^2 = f_0^2\bar{p}\chi'/N^2$ and integration in x give

$$\bar{p}u_0(f_0^2/N^2)B = \overline{p'w'} \qquad (8.12)$$

and evaluation of the integrals in equation 2.21 for a thin horizontal layer for the case of stationary flow gives

$$\bar{p}(du_0/dz)(f_0^2/N^2)B = d\,\overline{p'w'}/dz \qquad (8.13)$$

the first term on the right-hand side of (2.21) vanishing because of the phase shift in $\partial\chi'/\partial x$ and $\partial\chi'/\partial y$. Differentiation of (8.12) and substitution from (8.13) then lead to the result that $\bar{p}B/N^2$ and $p'w' - \rho Bu_0/N^2$ are independent of height if $u_0 \neq 0$. This result was first obtained by A. Eliassen who communicated it to the authors. It follows that equation 8.5, as well as the boundary and interface conditions, are homogeneous, and we may conclude that $\partial\chi_0/\partial t$ vanishes identically, i.e., that the second-order changes in the zonal flow are zero.

If the flow is not stationary, or there is horizontal shear in the undisturbed zonal current, or higher-order nonlinear interactions are taken into account, no such conclusion can be drawn. These would seem to be the interesting cases if one is to account for trapping when the thermally induced upper atmosphere circumpolar vortices are weak.

9. Application to the Upper Atmosphere

It follows from condition 3.6 that vertical energy propagation in standing waves in an atmosphere of uniform basic zonal velocity and temperature can occur when the velocity is positive but smaller than the modified Rossby critical velocity

$$U_c \equiv \beta/[(k^2 + l^2) + f_0^2/4H^2N^2]$$

In all other cases the waves are trapped. This criterion is modified but remains qualitatively applicable when the zonal velocity and the temperature vary. The Rossby critical velocity increases with the wavelengths of the disturbance in the zonal and meridional directions, and these are determined by wavelengths of the exciting forces. The principal Fourier component in the spectrum of the northern hemisphere topography has an azimuthal (longitudinal) wave number of about 2, corresponding to two continents and two oceans, and a meridional (latitudinal) wave number somewhat greater than 2. Hence, if we define $L_x \equiv 2\pi/k$ and $L_y \equiv 2\pi/l$, $L_x \leq \pi a \cos\phi_0 = 14{,}000$ km at 45°, and $L_y < \pi a = 20{,}000$ km. Taking $N^2 = 4 \times 10^{-4}$ sec^{-2}, corresponding to an isothermal atmosphere, we obtain max $U_c < 38$ m sec^{-1}. If the β-plane approximation is not made and the motion is expressed in spherical harmonics (see section 2), $k^2 + l^2$ is replaced by $n(n+1)/a^2$, where n is the degree of the spherical harmonic and a is the radius of the earth. The dominant n may be determined from the development of the earth's topography in spherical harmonics up to order 16 by *Prey* [1922]. He gives the coefficients a_n, a_{mn}, b_{mn} in the expansion of the nth-degree harmonic

$$Y_n = a_nP_n + \sum_{m=1}^{n}(a_{mn}\cos m\lambda + b_{mn}\sin m\lambda)P_n^m$$

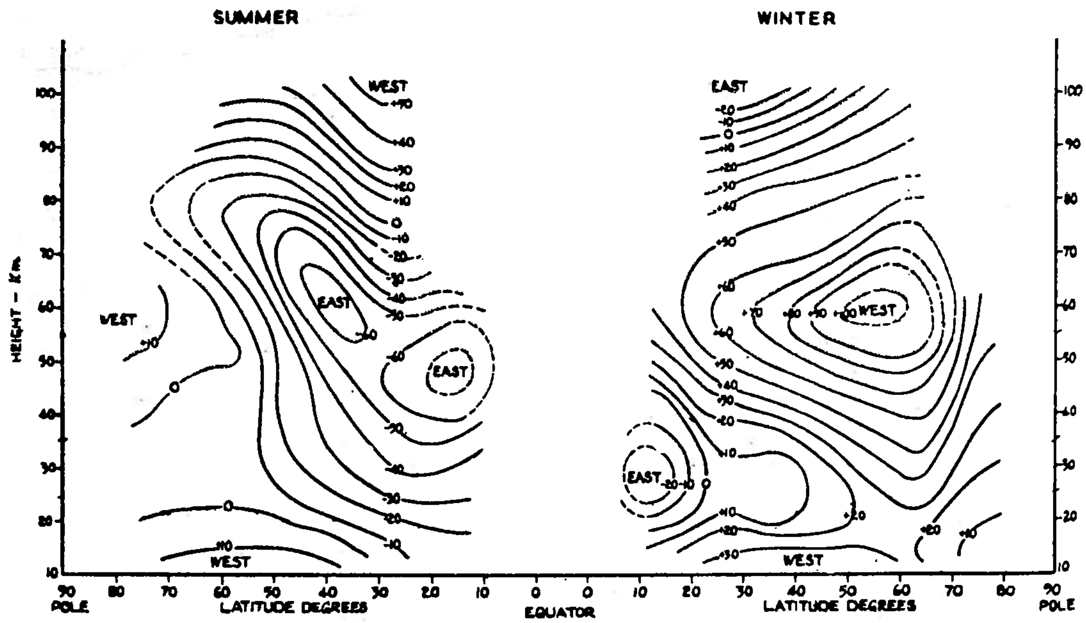

Fig. 1. Mean zonal winds in m sec^{-1} in the upper atmosphere for summer and winter (after Murgatroyd).

From these the root-mean-square values can be determined

$$(\overline{Y_n^2})^{1/2} = \left\{\frac{1}{2(2n+1)}\left[2a_n^2 + \sum_{m=1}^{n}\frac{(n+m)!}{(n-m)!}(a_{mn}^2 + b_{mn}^2)\right]\right\}^{1/2}$$

It is found that the dominant n is greater than 3, giving $U_c < 38$ m sec^{-1} as before. Hence one expects trapping when the mean zonal velocity exceeds approximately this value, or is negative.

The best available determination of the mean temperatures and zonal velocities in the upper atmosphere for winter and summer has been given by *Murgatroyd* [1957] from a critical review of all sources of data that were in existence at the time of his study. Mean temperatures and zonal velocities between the ground and 30 km have been given by *Kochanski* [1955] for the months January, April, July, and October along the 80°W meridian. Murgatroyd's and Kochanski's longitude-height sections of zonal velocity are reproduced in Figures 1 to 3. The profiles of temperature and N^2 averaged for the 30° to 60° latitude belt are shown in Figure 4. It is seen from Figures 1 and 2 that the planetary waves of middle latitudes cannot be expected to penetrate above about 20 km in summer and about 35 km in winter. Using Murgatroyd's wind and temperature data at high levels and Kochanski's at low levels, we have calculated ν^2 averaged between 30° and 60° from the general expression 3.2. The results are given in Figure 5 for $L \equiv (L_x^{-2} + L_y^{-2})^{-1/2} =$ 6000, 10,000, and 14,000 km, corresponding to $U_c = 13$, 31, and 49 m sec^{-1}, respectively. In summer ν^2 is negative above 20 km, and in winter it becomes negative above 30 km, but becomes positive again for $L = 14{,}000$ km above 55 km. Our qualitative conclusions are thus verified.

We note that there is a jump in ν^2 at the tropopause due to discontinuities in N^2 and du_0/dz. Wave reflection at a discontinuity was discussed in section 6, and a numerical example was worked out in section 7. In general a discontinuity between two layers transmits a large part of the energy unless the index of refraction in the upper layer is imaginary.

Since the winds in the upper troposphere and mesosphere reverse in direction from summer to winter, we might look for intermediate periods in which transmission into these regions becomes possible. *Batten* [1960] has prepared a time section of the zonal wind speed from rocket observations taken in the western United States at a number of stations between 30° and 40°N. His section is shown in Figure 6. A composite mean zonal wind distribution for the four seasons at middle latitudes was compiled by averaging Murgatroyd's and Kochanski's data over the 30° to 60°N latitude belt to obtain the summer and winter curves, and by averaging Batten's data over the three-month periods March 1 to June 1 and September 1 to December 1 to obtain the spring and autumn curves. These are shown in Figure 7. The absence of appreciable zonal velocities in spring signifies that by then the winter circumpolar vortex has broken down into a number of cellular circulations. It is then no longer possible to treat the wave motions as disturbances of a zonal vortex, and values of ν^2 computed for spring would not be meaningful. The autumn period is dominated by the winter circumpolar cyclone, which becomes highly developed in the latter half. However, as may be seen from Figure 7, the zonal winds are not as swift as they later become. Probably for this reason the computation of ν^2 for autumn, which is shown in Figure 8, does indicate the possibility of vertical energy propagation into the stratosphere. The figure shows that energy which tunnels through the layer between 10 and 20 km

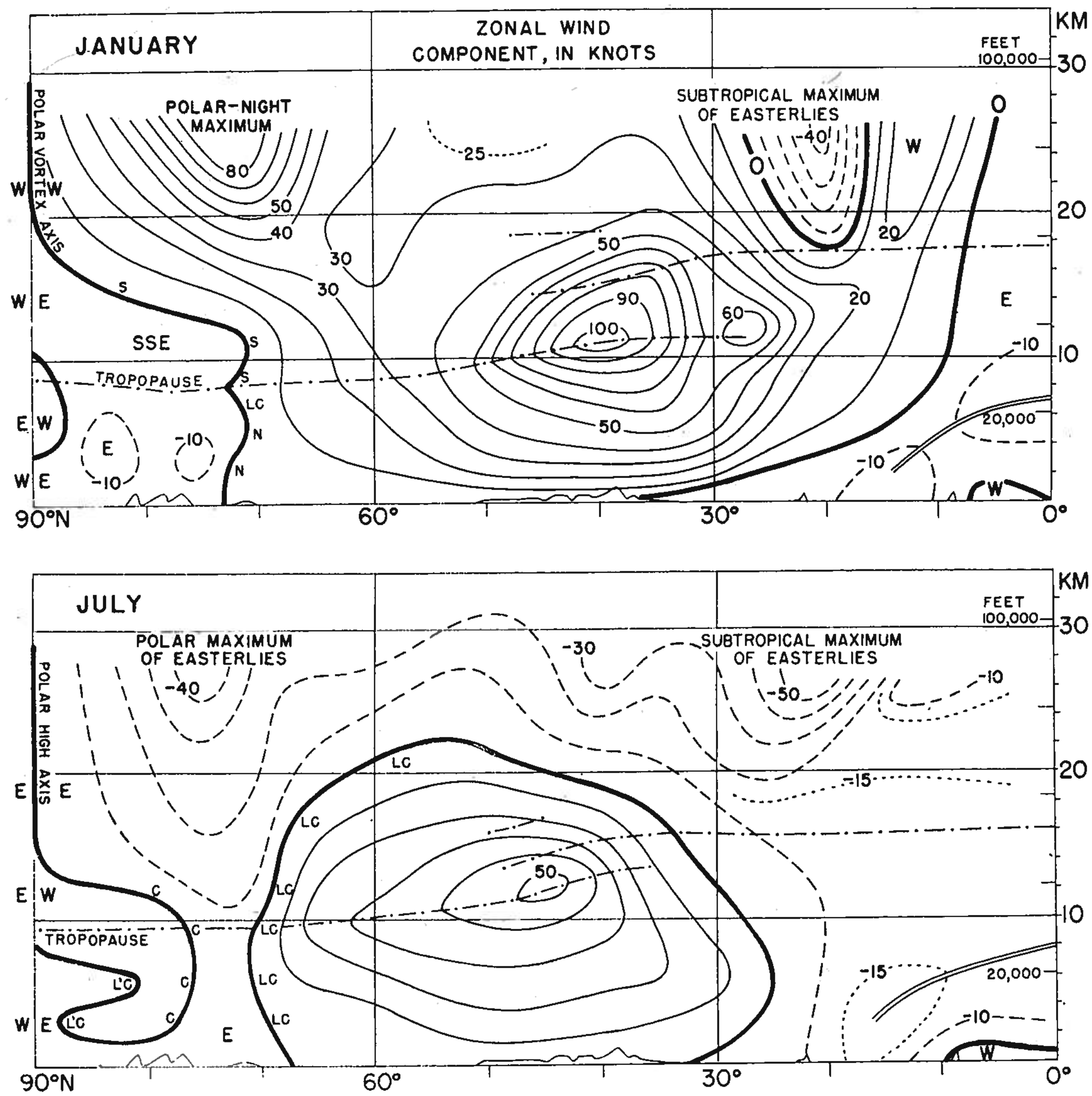

Fig. 2. Mean zonal winds in knots between 0 and 30 km for January and July along the 80°W meridian (after Kochanski).

will propagate to perhaps 60 km but will then be reflected by the thick trapping layer above that height. If energy could exist at $L = 14{,}000$ km, it would penetrate to 100 km, but such wavelengths are unrealistically large.

So far no catastrophic changes in the upper atmosphere have been reported. If large quantities of energy were actually to penetrate into the rarefied upper atmosphere, strong nonlinear interactions could occur which might modify the upper atmosphere wind and temperature structure in such a way as to insulate it against further energy flux. It is remarkable that the models so far considered (in which the undisturbed zonal velocity is a function of height only) permit no such interactions (section 8). It is impossible at present to say whether other models would give very different results. More calculations with more realistic models are needed to assess the importance of the nonlinear interactions.

The foregoing results are qualitatively in accord with what is known about the motions in the upper stratosphere. (The United States Weather Bureau 10-mb synoptic charts for the IGY period, July 1957 through June 1958, its

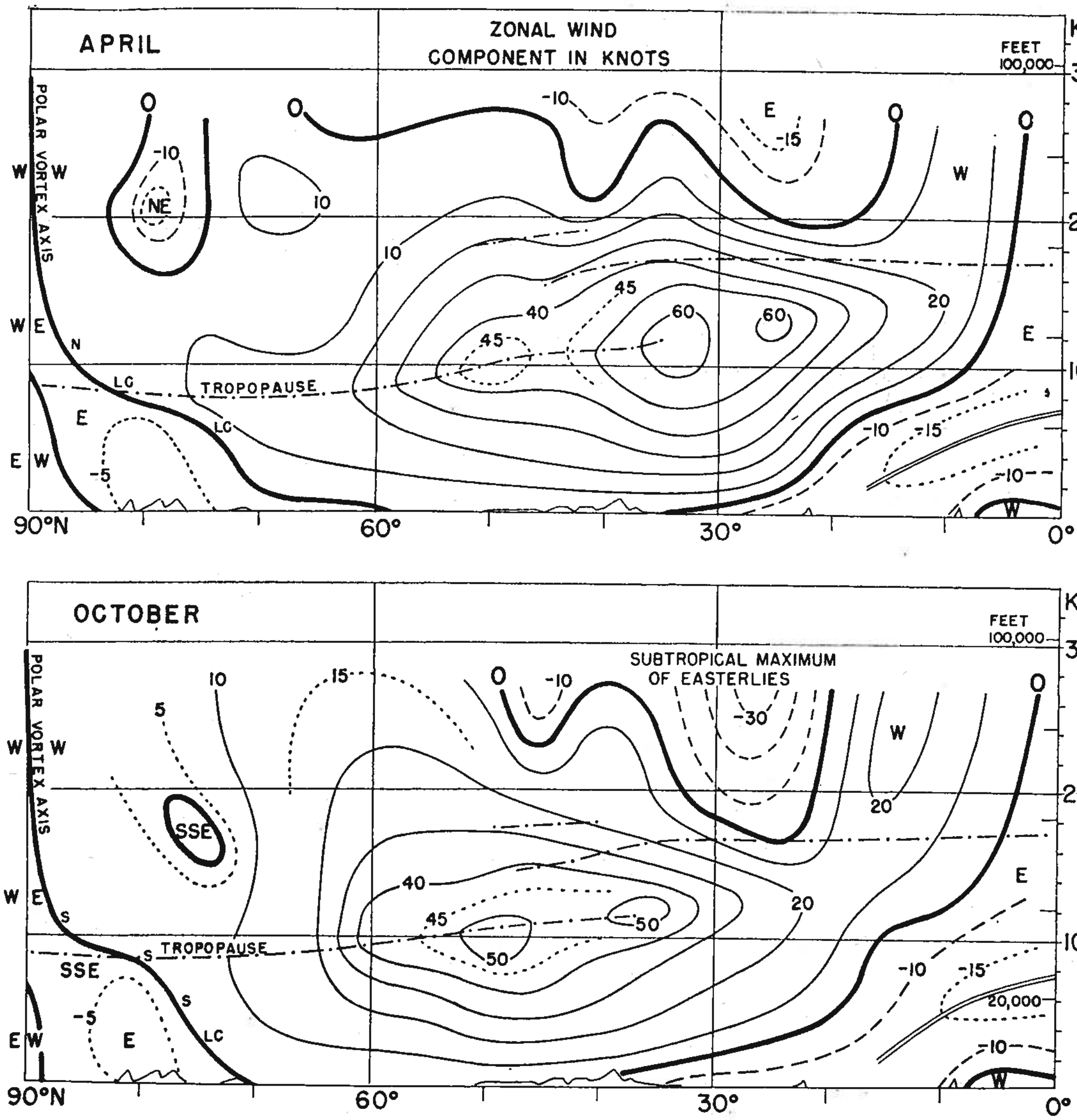

Fig. 3. Mean zonal winds in knots between 0 and 30 km for April and October along the 80°W meridian (after Kochanski).

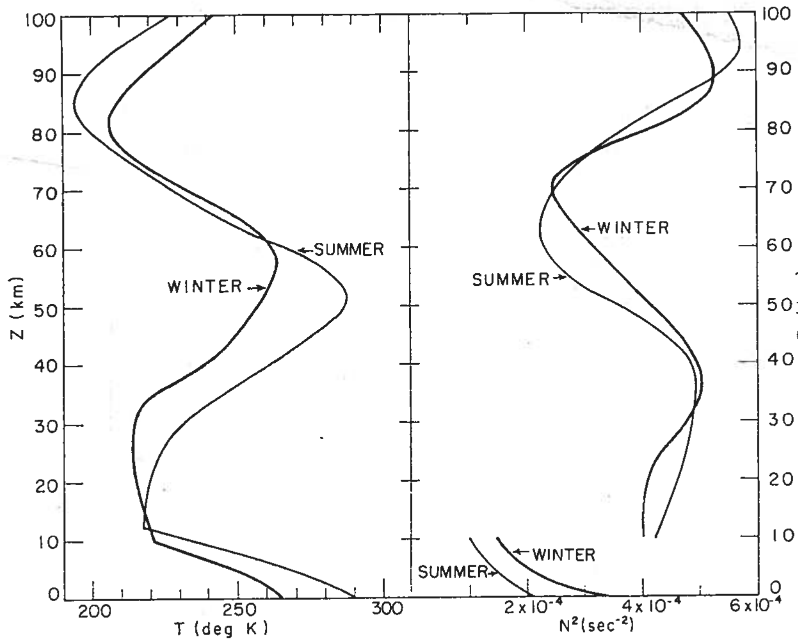

Fig. 4. Mean temperature and mean square Brunt-Vaisälä frequency in the upper atmosphere for summer and winter, averaged between 30° and 60°N.

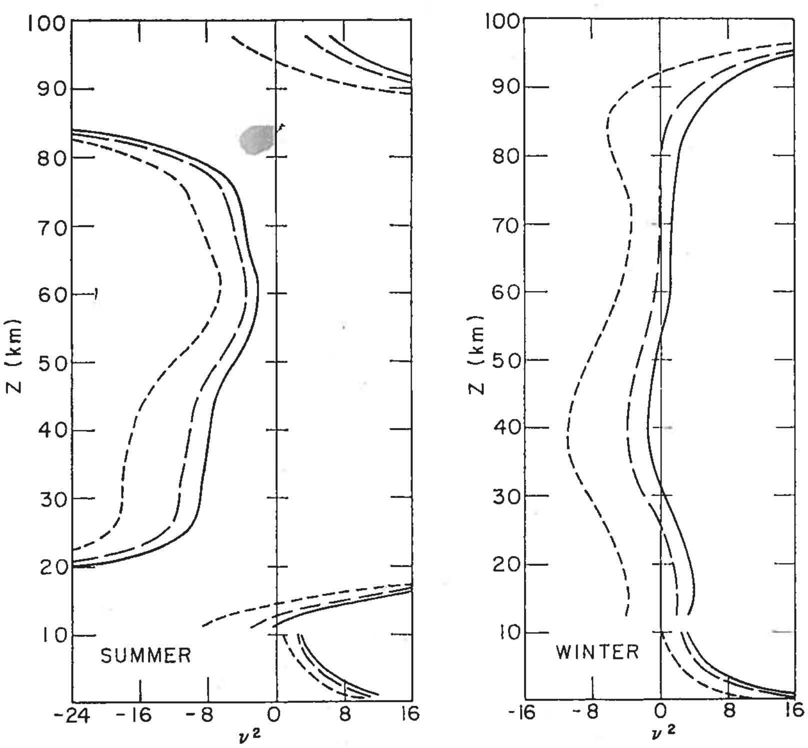

Fig. 5. The square of the index of refraction for summer and winter, averaged between 30° and 60°N. The short-dashed lines correspond to $L = 6{,}000$ km, the long-dashed lines correspond to $L = 10{,}000$ km, and the solid lines correspond to $L = 14{,}000$ km.

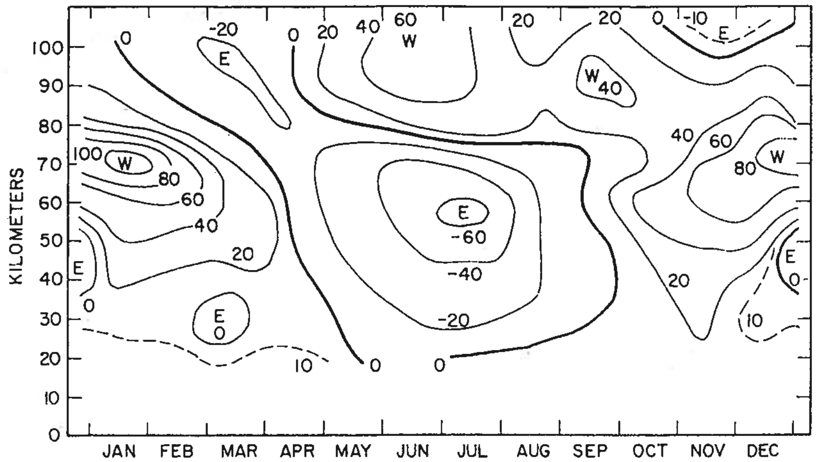

Fig. 6. Vertical time section of zonal winds in the upper atmosphere between latitudes 30° and 40°N (after Batten).

100-, 50-, and 30-mb charts for July 1957, and the daily 25-mb charts issued by the Institute for Meteorology and Geophysics of the Free University of Berlin for the period January through March 1958 are available; together with tropospheric charts, they give an indication of the links between the upper stratospheric, lower stratospheric, and tropospheric circulations.) The circulation in the winter hemisphere is dominated by a strong cyclonic circumpolar vortex which at times 'breaks down' and forms large meanders and cutoff vortices. When the westerly winds are strong there is no obvious connection between the upper stratospheric motions and those in the troposphere and lower stratosphere. The breakdown itself appears to be due to some form of instability—not baroclinic, perhaps, but barotropic [cf. *Murray*, 1960]—rather than to an interaction with the lower atmosphere. After the last breakdown, in

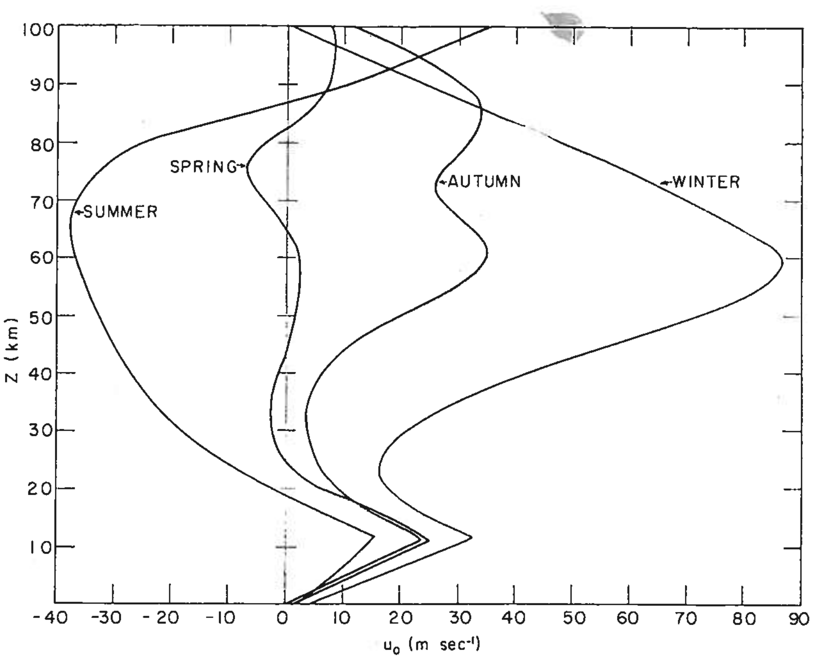

Fig. 7. Composite curves showing seasonal variation of mean zonal winds in middle latitudes up to 100 km.

March or April, the cyclonic vortex becomes progressively weaker, and, at that time, especially in lower latitudes, there does appear to be a connection between the upper stratospheric motions and the lower stratospheric and tropospheric motions; they are similar in scale and move together. As the summer advances, an anticyclonic circumpolar vortex is formed, which reaches a maximum intensity in July and then diminishes until September, when it is quickly replaced by the winter cyclonic vortex. The summer anticyclone is remarkably steady and impervious to all influences from below; the low-level influences do not reappear until autumn and then only for a short time. The qualitative evidence is thus in accord with our general conclusions.

An examination of Batten's time cross section (Figure 6) shows that the periods during which appreciable lower-upper atmosphere interaction can occur are in any case short. This observation has an important bearing on the problem of the general circulation of the upper atmosphere. It implies that during most of the year the planetary-wave interactions between the lower and upper atmospheres can be ignored. The circulations of the two parts would thus appear to be self-contained except so far as they interact through their axially symmetric components, but from considerations of the independence of the energy sources for the two circulations it would be surmised that this interaction, too, is small.

In summary, we conclude that the escape of large amounts of planetary-wave energy from the troposphere into the upper atmosphere is prevented throughout most of the year by the easterly or large westerly zonal winds above the tropopause. If propagation does take place, it must be during the spring or for a brief period in the autumn. But even then there is apparently enough trapping to prevent the occurrence of an atmospheric corona at the dissipative levels. In any event the upper atmosphere acts as a selective short-wave filter so that the long planetary waves produced by forced flow over continents or by differential heating over continents and oceans are more likely to penetrate to high levels than the shorter waves generated spontaneously by baroclinic instability. Since in most of the year planetary-wave energy is trapped in the lower stratosphere, the circulation in the upper stratosphere and mesosphere is, to a large extent, mechanically independent of the motion in the lower atmosphere.

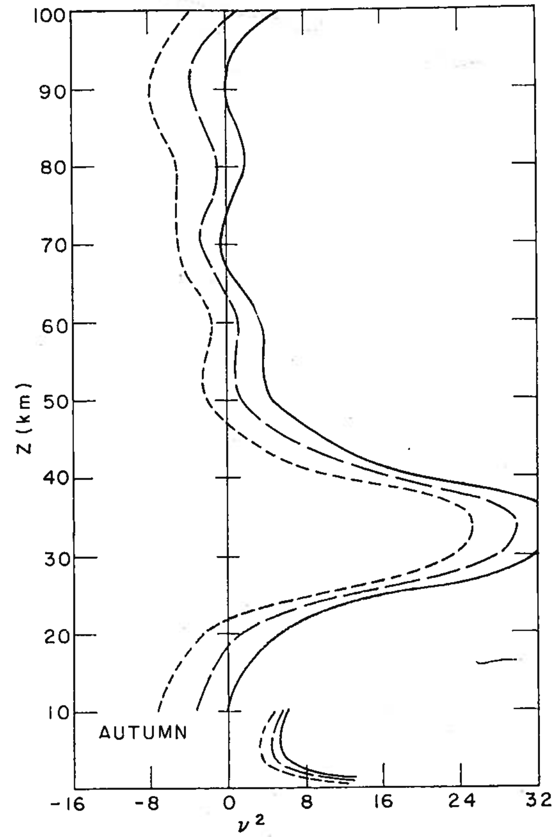

Fig. 8. The square of the index of refraction for autumn, averaged for the period September 1 to December 1. See Figure 5 for meaning of lines.

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Appendix

Boundary condition on a mountain. Suppose there is a rigid surface with equation

$$z = h(x, y)$$

(the origin of height being chosen so that the horizontal mean of h is zero). The boundary condition that this is a streamline is

$$w = u\,\partial h/\partial x + v\,\partial h/\partial y \qquad (z = h)$$

i.e.,

$$w + h\,\partial w/\partial z + \frac{1}{2!}h^2\,\partial^2 w/\partial z^2 + \cdots$$
$$= (u + h\,\partial u/\partial z + \cdots)\,\partial h/\partial x$$
$$+ (v + h\,\partial v/\partial z + \cdots)\,\partial h/\partial y \quad (z = 0)$$

by Taylor's theorem. Now we have taken $\mathbf{u} = (u_0 + u', v', w')$ as a superposition of a mean zonal flow and a small perturbation due to the mountain. Therefore

$$w' + h\,\partial w'/\partial z + \cdots = (u_0 + h\,du_0/dz$$
$$+ \cdots + u' + \cdots)\,\partial h/\partial x$$
$$+ (v' + \cdots)\,\partial h/\partial y \qquad (z = 0)$$

If $u_0(0) \neq 0$, it can be seen that $\mathbf{u}'$ is of order h and

$$w' = u_0\,\partial h/\partial x$$

on neglect of terms of order h^2. If h is absolutely integrable and of uniformly bounded variation in x, y, it has a real Fourier transform $h_0(k, l)$ and complex representation

$$h = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_0 e^{i(kx+ly)}\,dk\,dl$$

In this case all the equations and boundary conditions are linear, and so all wave components are dynamically independent and it is possible to confine our attention to a typical component. Thus we may put

$$h = h_0 e^{ikx}\cos ly$$

and use the boundary condition

$$w' = iku_0(0)h_0 e^{ikx}\cos ly \qquad (z = 0)$$

The linearization of the boundary condition has been implicitly based on the assumption that h is small for given functions $\bar{\rho}$, N^2, u_0 with $h\,\partial w'/\partial z \ll w'$, etc. If $u_0(0) \sim h(du_0/dz)_{z=0}$, nonlinear interaction of surface components occurs. This is illustrated below, where it is supposed that h is small for given $u_0(z)$ with $u_0(0) = 0$, $(du_0/dz)_{z=0} \neq 0$. Then $\mathbf{u}'$ is of order h^2, and the boundary condition becomes

$$w' = (du_0/dz)_{z=0}h\,\partial h/\partial x$$

on neglect of terms of order h^3. If

$$h\,\partial h/\partial x = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_0(k', l')e^{i(k'x+l'y)}\,dk'\,dl'$$

we can treat components separately by using the condition

$$w' = (du_0/dz)_{z=0}\,g_0\,e^{ikx}\cos^2 ly$$

Though g_0 is related to h_0 by Parseval's theorem and is not easy to derive in general, we can calculate g_0 directly if h is purely sinusoidal, i.e. if $h = h_0e^{i(kx+ly)}$ is the only surface component. Then

$$h\,\partial h/\partial x = Re h_0e^{ikx} Re ikh_0e^{ikx}\cos^2 ly$$
$$= -\tfrac{1}{2}kh_0^2\sin 2kx\cos^2 ly$$

from which g_0 may be obtained. Thus the boundary condition in this case is

$$w' = \tfrac{1}{2}ikh_0^2\left(\frac{du_0}{dz}\right)_{z=0}e^{2ikx}\cos^2 ly$$

Note that a sinusoidal mountain excites atmospheric waves of half its length by this means.

In the above case there is interaction of the harmonic components of the surface elevation in producing baroclinic waves, because the boundary condition is quadratic in h. However, the condition is linear in $\mathbf{u}'$, so there is no interaction of the baroclinic waves themselves.

In fact, atmospheric measurements give $u_0(0) \sim 5$ m sec^{-1} and $(du_0/dz)_{z=0} \sim 3$ m sec^{-1} km^{-1}, and so a complicated mixture of both conditions should be used for mountain heights of the order of 5/3 km. Since this is just the order of heights that mountains have, strictly it is not possible to use a linear theory at all. The rigorously required nonlinear boundary condition seems intractable, but it is also unwarranted in view of other approximations we have made (e.g., the assumption that the height of the mountain is much less than the height of the tropopause, and the linearization of the equations of flow). We aim to do no more than construct a consistent model close enough to the real atmosphere to estimate the nature and magnitude of vertical energy propagation. This may be done by taking

$$w' = W_0 e^{i(kx+ly)}$$

a linear boundary condition, which may be no more than a definition of $W_0(k, l)$ as a Fourier component of $w_{z=0}'$. This obviates the problem of origin of the motion and can represent any surface excitation, be it linear or nonlinear mountain waves, or differential surface heating.

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